Independence of Variations to Kruskal's Theorem in ACA_0

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Abstract

This paper will provide the explicit proofs of the independence from ACA_0 of four propositions that were set out by Smith[2]. The propositions express the fact that particular subsets of the set of all trees $\mathcal{T}$ are well–quasi–orderings. To prove the independence, the approach of this paper is to proceed from the assumption that $\mathrm{ACA}_0 \nvdash \mathrm{CWF}(\varepsilon_0)$[3] via the explicit formulation of a function $\psi : \varepsilon_0 \to A$, where $\mathcal{T} \supset A$ is the set of trees under consideration, and $\psi(\alpha) \trianglelefteq \psi(\beta) \Rightarrow \alpha \leq \beta$ — where $\trianglelefteq$ denotes homeomorphic embedding, which is specified to be, depending on the proposition, structure preserving or not.

1 Introduction

We will take $(A, \leq)$ is a *well–quasi ordering* if $(A, \leq)$ is a quasi–ordering (that is, it is reflexive and transitive) and the proposition $\mathrm{WQ}(A)$ is true, where $\mathrm{WQ}(A) \equiv \forall F : \mathbb{N} \to A, \exists i, j(i < j \wedge F(i) \leq F(j))$.

The following four well–quasi orderings are under consideration:

1. $\mathcal{B}$ is the set of all binary trees, where $\mathcal{B} \ni t_1 \trianglelefteq t_2 \in \mathcal{B}$ iff there is a homeomorphic (infimum–preserving) embedding of t_1 into t_2.

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2. $\mathcal{B}$ is the set of all binary structured trees, where $\mathcal{B} \ni t_1 \trianglelefteq t_2 \in \mathcal{B}$ iff there is a homeomorphic structure–preserving embedding of t_1 into t_2.

3. $\mathcal{B}_2$ is the set of all exactly binary trees, where $\mathcal{B}_2 \ni t_1 \trianglelefteq t_2 \in \mathcal{B}_2$ iff there is a homeomorphic embedding.

4. For each n, $\mathcal{Q}_n$ is the set of trees of height n. Again, $\mathcal{Q}_n \ni t_1 \trianglelefteq t_2 \in \mathcal{Q}_n$ iff there is a homeomorphic embedding.

In the first three cases, one is concerned with a well–quasi ordering $(A, \trianglelefteq)$, but it is not possible in ACA_0 and PA to prove this fact. The argument towards this will proceed from the assumption that[3]

$$\mathrm{ACA}_0 \nvdash \mathrm{CWF}(\varepsilon_0) \tag{1}$$

Here $\mathrm{CWF}(A) \equiv \forall k \exists n \forall (\alpha_0, \ldots, \alpha_n)(\forall i \leq n(|t_i| \leq k + i \Rightarrow \exists i < j(\alpha_i \leq \alpha_j)))$.

Taking any function $F : \mathbb{N} \to A$ and considering the set of n–tuples $(F(0), \ldots, F(n))$ it is clear — and will be used in the argument — that the following proposition is implied:

$$\mathrm{ACA}_0 \nvdash \forall F : \mathbb{N} \to \varepsilon_0, \exists i, j(i < j \wedge F(i) \leq F(j)) \tag{2}$$

One will show that, for each of the first three propositions there exists an order–preserving function $\psi : \varepsilon_0 \to A$. From this fact it follows that $\mathrm{ACA}_0 \vdash \mathrm{WQ}(A)$ would imply that $\mathrm{ACA}_0 \vdash \mathrm{CWF}(\varepsilon_0)$, contrary to the established result.

In the fourth case, what is to be proven is that $\mathrm{ACA}_0 \nvdash \forall n \mathrm{WQ}(\mathcal{Q}_n)$. The argument proceeds similarly. It is shown that if $\mathrm{ACA}_0 \vdash \forall n \mathrm{WQ}(\mathcal{Q}_n)$ then $\mathrm{ACA}_0 \vdash \mathrm{CWF}(\varepsilon_0)$ and thus the former cannot hold.

The proofs will not be given in the original order.

2 Preliminaries

Definition 2.1. $\omega_0 \doteq \omega^0 \qquad \omega_{s(n)} \doteq \omega^{\omega_n}$.

Definition 2.2. $\varepsilon_0 \doteq \min\{\xi \in \mathrm{On} | \xi = \omega^\xi\}$.

Definition 2.3. $I_m \doteq \{n \in \mathbb{N} | n \leq m\}$. Clearly $|I_m| = m + 1$.

Definition 2.4. For any function f and any sets A and B such that $f : A \to B$, the notation $f[A]$ is taken to refer to the set $\{x \in B | \exists a \in A(f(a) = x)\}$.

Definition 2.5.

$$\begin{array}{rrcl} & A^0 & \doteq & \{\emptyset\} \\ & A^1 & \doteq & A \\ \forall n > 1 & A^n & \doteq & A \times A^{n-1} \end{array}$$

Lemma 2.1. *The following facts about ordinals are used[1], for all $\alpha, \beta, \gamma, \delta$:*

- $\alpha \geq \beta \wedge \gamma \geq \delta \Rightarrow \alpha + \gamma \geq \beta + \delta$.
- *For* $\beta \neq 0$, $\alpha \leq \beta^\alpha$.
- $\alpha > \beta \Rightarrow \alpha + \gamma > \beta$.
- *For* $\gamma \geq 0$, $\alpha \leq \beta \Rightarrow \gamma^\alpha \leq \gamma^\beta$.
- $\alpha \leq \beta \Rightarrow \alpha \leq \omega^0 \cdot \beta$.

Cantor Normal Form (in base ω) Furthermore, it is assumed to be known that for each $\alpha < \varepsilon_0$, it holds that either $\alpha = 0$ or $\exists! \alpha_0 \geq \ldots \geq \alpha_n (\alpha = \omega^{\alpha_0} + \ldots + \omega^{\alpha_n})$.

Lemma 2.2. *The following holds for the Normal Form expansion of $\alpha > 0$, where the coefficients in base ω are denoted by α_i,*

1. $\alpha_0 < \alpha$.
2. *If* $\alpha \in \omega_n$ *then* $\alpha_0, \ldots, \alpha_m \in \omega_{n-1}$.

Proof.

1. $0 < \alpha_0 < \varepsilon_0$ and thus $\omega^{\alpha_0} > \alpha_0$. If not $\alpha_0 < \alpha$ then $\omega^{\alpha_0} > \alpha_0 \geq \alpha$, absurd.
2. If for any $i \leq m$, $\alpha_i > \omega_{n-1}$ then $\alpha \geq \omega^{\alpha_i} > \omega^{\omega_{n-1}} = \omega_n$, contrary to the assumption.

□

Also, $a \# b$ will be taken to denote the natural (Hessenberg) sum. It has the property that if $a_0 \geq \ldots \geq a_n > 0$ and $\forall i \forall \delta, \gamma < a_i (\delta + \gamma < a_i)$, then $a_0 + \ldots + a_n = a_0 \# \ldots \# a_n$[1]. In particular, this holds for Cantor Normal Form expansions in the basis of ω.

Furthermore, if $\alpha < \omega_n$ then $\alpha_0, \ldots, \alpha_m \in \omega_{n-1}$.

3 Trees

Definition 3.1. Let $\mathcal{T}$ denote the set of all trees. Let 0 be used to refer to the trivial tree consisting only in a root. Let there be for each $n \in \mathbb{N}$ an injective function $\bullet_n : \mathcal{T}^{n+1} \to \mathcal{T}$ such that for $t_0, \ldots, t_n \in \mathcal{T}$, the tree consisting of a root whose successors are $t_0, \ldots, t_n$ is referred to by $\bullet_n(t_0, \ldots, t_n)$. The set of all trees $\mathcal{T}$ is defined as the smallest set $\mathcal{T}$ such that

$$\begin{array}{rcl} 0 & \in & \mathcal{T} \\ \forall n \in \mathbb{N} \quad t_0, \ldots, t_n & \in & \mathcal{T} \Rightarrow \bullet_n(t_0, \ldots, t_n) \in \mathcal{T} \end{array}$$

In the following there will be dealt with subsets of $\mathcal{T}$.

4 Proposition ii

Let $\mathcal{B} \subset \mathcal{T}$ be the set of all binary structured trees, $0 \in \mathcal{B}$ the trivial tree consisting only of a root and $\bullet : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, injective, where $\bullet(a, b)$ yields the tree that has as the two branches of the root the trees a and b.

We assume that $\forall a, b(0 \neq \bullet(a, b))$.

We define the following relation on the set $\mathcal{B}$ to represent direct embeddability.

Definition 4.1. $\trianglelefteq^-$ is the smallest possible relation such that:

$$\begin{array}{rll} \forall t \in \mathcal{B}_2 & 0 \trianglelefteq^- t & \\ \forall s_1, s_2, t_1, t_2 \in \mathcal{B}_2 & \bullet(s_1, s_2) \trianglelefteq^- \bullet(t_1, t_2) \Leftrightarrow & (\bullet(s_1, s_2) \trianglelefteq^- t_1) \vee \\ & & (\bullet(s_1, s_2) \trianglelefteq^- t_2) \vee \\ & & (s_1 \trianglelefteq^- t_1 \wedge s_2 \trianglelefteq^- t_2) \end{array}$$

Lemma 4.1. $\forall a(a \trianglelefteq^- a)$

Proof. By induction.

- **Basis** $a = 0$. Follows by definition.
- **Inductive** $a = \bullet(n_1, n_2)$. $n_i = n_i$, $i \in \{1, 2\}$, thus by the inductive hypothesis, $n_i \trianglelefteq n_i$. Thus, by definition of $\trianglelefteq^-$, $a \trianglelefteq^- a$.

□

Definition 4.2. $\trianglelefteq$ is the transitive closure of $\trianglelefteq^-$, which means that it holds that $a \trianglelefteq b \Leftrightarrow \exists n \exists a_0, a_1, \ldots, a_n (a = a_0 \trianglelefteq^- a_1 \trianglelefteq^- \ldots \trianglelefteq^- a_n = b)$.

Obviously, $\trianglelefteq$ is a quasi–ordering.

Lemma 4.2. *Let $(A, \trianglelefteq)$ be the transitive closure of $(A, \trianglelefteq^-)$. Let $\forall a(a \trianglelefteq^- 0 \Rightarrow a = 0)$. Then $\forall a(a \trianglelefteq 0 \Rightarrow a = 0)$.*

Proof. $t \trianglelefteq 0 \overset{\text{def}}{\Leftrightarrow} \exists n \exists a_0, \ldots, a_n . t = a_0 \trianglelefteq^- \ldots \trianglelefteq^- a_n = 0$. Proof by induction on n.

- **Basis** $n = 0$ is trivial, since the right hand side above shows $t = 0$. For $n = 1$: $t \trianglelefteq^- 0$. Thus $t = 0$.

- **Inductive** $n > 0$: $\exists a_0, \ldots, a_{n-1}, a_n (t = a_0 \trianglelefteq^- \ldots \trianglelefteq^- a_{n-1} \trianglelefteq^- a_n = 0$. From $a_{n-1} \trianglelefteq^- 0$ follows that $a_{n-1} = 0$, as before. Then, by the inductive hypothesis, $t = a_0 = 0$.

□

Corollary 4.3. *0 is a minimal element of $(\mathcal{B}, \trianglelefteq)$.*

Proof. The assumption is obvious from the fact that $\trianglelefteq^-$ is the minimal relation for which the conditions mentioned above hold. □

I then define the set $\mathbb{T}_{\{\phi,0\}}$ as the smallest set of terms, such that

$$0 \in \mathbb{T}_{\{\phi,0\}} \tag{3}$$

$$a, b \in \mathbb{T}_{\{\phi,0\}} \Rightarrow \phi(a, b) \in \mathbb{T}_{\{\phi,0\}} \tag{4}$$

Every term has an ordinal as its interpretation. The interpretation of $0 \in \mathbb{T}_{\{\phi,0\}}$ is $0 \in \text{On}$, and the interpretation of $\phi(\alpha, \beta)$ is $\omega^\alpha + \beta$.

By *Cantor's Normal Form*,

$$\forall \alpha \neq 0, \alpha < \varepsilon_0 (\exists! \alpha_1 \geq \ldots \geq \alpha_n (\alpha = \omega^{\alpha_1} + (\ldots + \omega^{\alpha_n}))) \tag{5}$$

One writes $\alpha =_N (\beta, \gamma)$ if $\alpha \neq 0$ and, the Cantor Normal Form expansion of α in the basis of ω being $\alpha = \omega^{\alpha_0} + \ldots + \omega^{\alpha_n}$, $\beta = \alpha_0$ and $\gamma = \sum_{i=1}^n \omega^{\alpha_i}$. We define a function $\phi : \text{On} \times \text{On} \to \text{On}$ as $\phi(\alpha, \beta) = \omega^\alpha + \beta$.

Lemma 4.4. *For each $0 < \alpha < \varepsilon_0$ there exist unique β and γ such that $\alpha =_N (\beta, \gamma)$. Also, $\alpha = \phi(\beta, \gamma)$ and $\beta < \alpha$ and $\gamma < \alpha$.*

Proof. The existence and uniqueness is obvious from the Cantor Normal Form. The equation $\alpha = \phi(\beta, \gamma)$ arises from the definition of ϕ. After lemma 2.2 $\beta < \alpha$. For the last fact, if $\gamma > \alpha$ then, by lemma 2.1, $\omega^\beta + \gamma > \alpha$, absurd. If $\gamma = \alpha$, then, since $\gamma > 0$, there exist $\gamma_0 \geq \ldots \geq \gamma_n$ such that $\gamma = \omega^{\gamma_0} + \ldots + \omega^{\gamma_n}$. Then $\alpha = \omega^\beta + \omega^{\gamma_0} + \ldots + \omega^{\gamma_n} = \gamma = \omega^{\gamma_0} + \ldots + \omega^{\gamma_n}$, which contradicts the uniqueness of the Cantor Normal Form. □

Corollary 4.5. *For each $\alpha < \varepsilon_0$ there is a term $t_\alpha \in \mathbb{T}_{\{\phi,0\}}$ such that the ordinal that is its interpretation is α.*

Proof. By induction on α. If $\alpha = 0$, then $t_\alpha = 0$ and its interpretation is by definition 0. If $\alpha > 0$, then by lemma 4.4, there exist unique β, γ such that $\alpha =_N (\beta, \gamma)$, and $\beta, \gamma < \alpha$. By the inductive hypothesis, it can thus be assumed that there exist t_β and t_γ whose interpretations are β and γ, respectively. Then $t_\alpha = \phi(t_\beta, t_\gamma)$ and its interpretation is α. □

Lemma 4.6. *For any $\alpha, \beta < \varepsilon_0$, if $\alpha =_N (\alpha_1, \alpha_2)$ and $\beta =_N (\beta_1, \beta_2)$, then*

1. $\alpha \leq \beta_1 \Rightarrow \alpha \leq \phi(\beta_1, \beta_2)$
2. $\alpha \leq \beta_2 \Rightarrow \alpha \leq \phi(\beta_1, \beta_2)$
3. $\alpha_1 \leq \beta_1 \wedge \alpha_2 \leq \beta_2 \Rightarrow \phi(\alpha_1, \alpha_2) \leq \phi(\beta_1, \beta_2)$.

Proof. Due to lemma 4.4, $\alpha = \phi(\alpha_1, \alpha_2)$ and $\beta = \phi(\beta_1, \beta_2)$.

1. $\alpha \leq \beta_1$. Thus $\alpha \leq \omega^{\beta_1} \leq \omega^{\beta_1} + \beta_2 = \phi(\beta_1, \beta_2)$.
2. $\alpha \leq \beta_2 \leq \omega^{\beta_1} + \beta_2 \leq \beta$.
3. $\alpha_1 \leq \beta_1$ and thus $\omega^{\alpha_1} \leq \omega^{\beta_1}$. Then, by lemma 2.1 $\phi(\alpha_1, \alpha_2) \leq \phi(\beta_1, \beta_2)$.

□

Definition 4.3. One defines a function $\psi : \varepsilon_0 \to \mathcal{B}$ as follows, and will then show that the property $\psi(\alpha) \trianglelefteq \psi(\beta) \Rightarrow \alpha \leq \beta$ holds. For each $\alpha \in \varepsilon_0$, one finds the term $t_\alpha \in \mathbb{T}_{\{\phi,0\}}$ and associates with it a tree inductively:

$$\begin{aligned} \psi(0) &\doteq 0 \\ \psi(\phi(a, b)) &\doteq \bullet(\psi(a), \psi(b)) \end{aligned} \tag{6}$$

Lemma 4.7. $\forall \alpha (\psi(\alpha) = 0 \Leftrightarrow \alpha = 0)$

Proof. $\Leftarrow$: By definition of ψ.
$\Rightarrow$: If $\alpha \neq 0$, then $\alpha = \phi(\alpha_1, \alpha_2)$, and $\psi(\alpha) = \bullet(\psi(\alpha_1), \psi(\alpha_2))) \neq 0$. □

Lemma 4.8. *$\psi(\alpha) = \psi(\beta) \Rightarrow \alpha = \beta$ (ψ is injective)*

Proof. Proof by induction on the complexity of $\psi(\alpha) \in \mathbb{T}_{\{\phi,0\}}$.

- **Basis** $\psi(\alpha) = \psi(\beta) = 0$. By lemma 4.7, $0 = \alpha = \beta$.
- **Inductive** $\psi(\alpha) \neq 0$. By lemma 4.7, $\alpha \neq 0 \neq \beta$. Thus $\alpha = \phi(\alpha_1, \alpha_2)$ and $\beta = \phi(\beta_1, \beta_2)$. Then $\bullet(\psi(\alpha_1), \psi(\alpha_2)) = \bullet(\psi(\beta_1), \psi(\beta_2))$. By the injectivity of the function $\bullet$ we find that $\psi(\alpha_1) = \psi(\beta_1)$ and $\psi(\alpha_2) = \psi(\beta_2)$, and as a result of the inductive hypothesis $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, whence $\alpha = \beta$.

□

Lemma 4.9. *$t \trianglelefteq^- \psi(\beta) \Rightarrow \exists\alpha(\psi(\alpha) = t)$*

Proof. By induction on the complexity of $\psi(\beta)$.

- **Basis** $\psi(\beta) = 0$. By lemma 4.7 $t = 0$. By the definition of ψ, $\psi(0) = 0$ and thus $\exists\alpha(\psi(\alpha) = t)$.
- **Inductive** $\psi(\beta) \neq 0$. If $t = 0$ then $\psi(0) = t$. Let us assume $t \neq 0$, which implies $t = \bullet(t_1, t_2)$. Then by lemma 4.7 $\beta \neq 0$. Thus $\beta = \phi(\beta_1, \beta_2)$. Then $\psi(\beta) = \bullet(\psi(\beta_1), \psi(\beta_2))$. By definition of $\trianglelefteq^-$ one of the following must hold:
  - $t \trianglelefteq^- \psi(\beta_1)$. Then by the inductive hypothesis, $\exists\alpha(t = \psi(\alpha))$.
  - $t \trianglelefteq^- \psi(\beta_2)$. As above, $\exists\alpha(t = \psi(\alpha))$.
  - $t_1 \trianglelefteq^- \psi(\beta_1)$ and $t_2 \trianglelefteq^- \psi(\beta_2)$. By the inductive hypothesis, for $i = 1, 2$, $\exists\alpha_i(t_i = \phi(\alpha_i))$. Then $\psi(\phi(\alpha_1, \alpha_2)) = \bullet(\psi(\alpha_1), \psi(\alpha_2)) = \bullet(t_1, t_2) = t$ and thus $\exists\alpha(t = \psi(\alpha))$.

□

Lemma 4.10. *$\psi(\alpha) \trianglelefteq^- \psi(\beta) \Rightarrow \alpha \leq \beta$*

Proof. By induction on the complexity of $\psi(\beta)$:

- $\psi(\beta) = 0$. Then, by lemma 4.3 $\psi(\alpha) = 0$. By lemma 4.7 $\alpha = \beta = 0$. Thus in particular $\alpha \leq \beta$.

- $\psi(\beta) \neq 0$. Induction on the complexity of $\psi(\alpha)$:
  - $\psi(\alpha) = 0$. Then, by lemma 4.7 $\alpha = 0$ and thus for all β, $0 = \alpha \leq \beta$.
  - $\psi(\alpha) \neq 0$. Again, by lemma 4.7, $\alpha \neq 0, \beta \neq 0$. By the Cantor Normal Form, $\exists \alpha_1, \alpha_2(\alpha = \phi(\alpha_1, \alpha_2)$ and $\exists \beta_1, \beta_2(\beta = \phi(\beta_1, \beta_2)$. Then $\psi(\alpha) = \bullet(\psi(\alpha_1), \psi(\alpha_2))$ and $\psi(\beta) = \bullet(\psi(\beta_1), \psi(\beta_2))$.

    Then the premise is equivalent to $\bullet(\alpha_1, \alpha_2) \trianglelefteq^- \bullet(\beta_1, \beta_2)$.

    By the definition of $\trianglelefteq^-$, this implies that one of the following holds:
    * $\psi(\alpha) = \bullet(\psi(\alpha_1), \psi(\alpha_2)) \trianglelefteq^- \psi(\beta_1)$. By the inductive hypothesis $\alpha \leq \beta_1$. By lemma 4.6, $\alpha \leq \beta$.
    * $\psi(\alpha) = \bullet(\psi(\alpha_1), \psi(\alpha_2)) \trianglelefteq^- \psi(\beta_2)$. By the inductive hypothesis $\alpha \leq \beta_2$. By lemma 4.6, $\alpha \leq \beta$.
    * $\psi(\alpha_1) \trianglelefteq^- \psi(\beta_1)$ and $\psi(\alpha_2) \trianglelefteq^- \psi(\beta_2)$. By the inductive hypothesis $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$. Then by lemma 4.6, $\phi(\alpha_1, \alpha_2) \leq \phi(\beta_1, \beta_2)$, and thus $\alpha \leq \beta$.

□

Lemma 4.11. *Let $(A, \trianglelefteq)$ be the transitive closure of a quasi–ordering $(A, \trianglelefteq^-)$. Let $\psi : X \to A$, where $(X, \leq)$ forms a transitive relation, such that $\forall \alpha, \beta(\psi(\alpha) \trianglelefteq^- \psi(\beta) \Rightarrow \alpha \leq \beta)$ and $\forall a(a \trianglelefteq^- \psi(\beta) \Rightarrow \exists \alpha(\psi(\alpha) = a))$. Then $\forall \alpha, \beta(\psi(\alpha) \trianglelefteq \psi(\beta) \Rightarrow \alpha \leq \beta$.*

Proof. Let us assume that $\psi(\alpha) \trianglelefteq \psi(\beta)$.

This means that $\exists n \exists a_0, \ldots, a_n(\psi(\alpha) = a_0 \trianglelefteq^- \ldots \trianglelefteq^- a_n = \psi(\beta)$. Proof by induction.

- **Basis** $n = 0$. $\psi(\alpha) = \psi(\beta)$. Since $(A, \trianglelefteq^-)$ is a quasi–ordering, it follows that $\psi(\alpha) \trianglelefteq^- \psi(\beta)$, and thus by assumption $\alpha \leq \beta$. For $n = 1$ the premise is already $\psi(\alpha) \trianglelefteq^- \psi(\beta)$.

- **Inductive** $n > 1$. Thus $\exists n \exists a_0, \ldots, a_{n-1}, a_n(\psi(\alpha) = a_0 \trianglelefteq^- \ldots \trianglelefteq^- a_{n-1} \trianglelefteq^- a_n = \psi(\beta)$ By hypothesis $\exists \alpha_{n-1}(\psi(\alpha_{n-1}) = a_{n-1})$. Since $\psi(\alpha_{n-1}) \trianglelefteq^- \psi(\beta)$ by hypothesis $\alpha_{n-1} \leq \beta$. By the inductive hypothesis, $\alpha \leq \alpha_{n-1}$. Then, by the transitivity of $\leq$, $\alpha \leq \beta$.

□

Corollary 4.12. *There exists a function* $\psi : \varepsilon_0 \to \mathcal{B}$ *such that* $\psi(\alpha) \trianglelefteq \psi(\beta) \Rightarrow \alpha \leq \beta$.

Proof. The assumptions are lemma 4.10 and 4.9. $\square$

Theorem 4.13. *Let* $(A, \trianglelefteq)$ *be a quasi–ordering. If there exists a function* $\psi : \varepsilon_0 \to A$*, such that* $\psi(\alpha) \trianglelefteq \psi(\beta) \Rightarrow \alpha \leq \beta$*, then* $\mathrm{ACA}_0 \nvdash \mathrm{WQ}(A)$.

Proof. Since

$$\mathrm{ACA}_0 \nvdash \mathrm{CWF}(\varepsilon_0) \Rightarrow \mathrm{ACA}_0 \nvdash \forall F : \mathbb{N} \to \varepsilon_0, \exists i, j (i < j \land F(i) \leq F(j)) \quad (7)$$

Proof by absurdity. Let us assume that

$$\mathrm{ACA}_0 \vdash \mathrm{WQ}(A) \quad (8)$$

Let us take any function $G : \mathbb{N} \to \varepsilon_0$.

By hypothesis there exists at least a function $\psi : \varepsilon_0 \to A$ such that $\psi(\alpha) \trianglelefteq \psi(\beta) \Rightarrow \alpha \leq \beta$. Now $F \doteq \psi \circ G$. Thus $F : \mathbb{N} \to A$. By our assumption, equation 8, it must be that $\exists i, j (i < j \land F(i) \leq F(j))$. This means that $F(i) = \psi(G(i)) \leq \psi(G(j)) = F(j)$. Then $G(i) < G(j)$. The same argument can be repeated for any other function $G : \mathbb{N} \to \varepsilon_0$. Thus it must be that $\mathrm{ACA}_0 \vdash \mathrm{CWF}(\varepsilon_0)$, which is absurd. $\square$

Corollary 4.14. $\mathrm{ACA}_0 \nvdash \mathrm{WQ}(\mathcal{B})$

Proof. Clearly $\mathcal{B}$ is a quasi–ordering. The latter assumption is corollary 4.12. $\square$

5 Proposition iv

Definition 5.1. There is defined a function $d : \mathcal{T} \to \mathbb{N}$ to represent the depth of each tree. It is defined as follows.

$$\begin{aligned} d(0) &= 0 \\ \forall n \in \mathbb{N} \quad d(\bullet_n(t_0, \ldots, t_n)) &= \max\{d(t_0), \ldots, d(t_n)\} + 1 \end{aligned}$$

Let $\mathcal{Q}_n \doteq \{t \in \mathcal{T} | d(t) = n\}$ be the set of all trees of depth exactly n and $\mathcal{Q}_{\leq n} \doteq \{t \in \mathcal{T} | d(t) \leq n\}$ the set of all trees of height at most n, $0 \in \mathcal{Q}_0$ the trivial tree consisting only of a root and $\bullet_m : \mathcal{Q}_{n-1}^{m+1} \to \mathcal{Q}_n$, injective, where $\bullet_m(a_0, \ldots, a_m)$ yields the tree that has as branches of the root the trees $a_0, \ldots, a_n$.

We assume that $\forall m \in \mathbb{N} \forall a_0, \ldots, a_m (0 \neq \bullet(a_0, \ldots, a_m))$.

We define the following relation on $\mathcal{Q}_{<n}$ to represent direct embeddability.

Definition 5.2. $\trianglelefteq^-$ is the smallest possible relation in $\mathcal{Q}_{<n}$, such that:

1. $0 \trianglelefteq^- 0$
2. $\forall m \forall t_0, \ldots, t_m, \quad 0 \trianglelefteq^- \bullet_m(t_0, \ldots, t_m)$
3. $\forall m_t, m_s \forall t_0, \ldots, t_{m_t}, s_0, \ldots, s_{m_s}, \quad \bullet(s_0, \ldots, s_{m_s}) \trianglelefteq^- \bullet(t_0, \ldots, t_{m_t}) \Leftrightarrow$
   (a) $\exists i (s \trianglelefteq^- t_i) \quad \vee$
   (b) $\exists F : I_{m_s} \to I_{m_t} \left(\forall i, j \left(F(i) = F(j) \Rightarrow i = j \wedge s_i \trianglelefteq^- t_{F(i)} \right) \right)$

Lemma 5.1. $\forall a (a \trianglelefteq^- a)$

Proof. By induction.

- **Basis** $a = 0$. Follows by definition.
- **Inductive** $a = \bullet_m(a_0, \ldots, a_m)$. $a_i = a_i$, $i \in I_m$, thus by the inductive hypothesis, $a_i \trianglelefteq^- a_i$. Thus, by definition of $\trianglelefteq^-$, $a \trianglelefteq^- a$.

□

Lemma 5.2. *$a \trianglelefteq^- b \trianglelefteq^- c \Rightarrow a \trianglelefteq^- c$ ($\trianglelefteq^-$ is transitive.)*

Proof. By induction on the complexity of a and b and c. If $c = 0$ then $b \trianglelefteq^- c$ implies $b = 0$ and consequently $a = 0$. If $a = 0$ then obviously $a \trianglelefteq^- c$. Otherwise $a = \bullet_m(a_0, \ldots, a_m)$. Then obviously $b \neq 0$. Thus $b = \bullet_n(b_0, \ldots, b_n)$. Then also $c = \bullet_p(c_0, \ldots, c_p)$. $b \trianglelefteq^- c$ implies that one of the following holds:

- $b \trianglelefteq^- c_i$. By the inductive hypothesis, from $a \trianglelefteq^- b \trianglelefteq^- c_i$ follows $a \trianglelefteq^- c_i$. Thus, by the definition of $\trianglelefteq^-$ it follows that $a \trianglelefteq^- c$.
- There exists an injective $F : I_n \to I_p$ such that for all $i \leq n$, $b_i \trianglelefteq^- c_{F(i)}$. We distinguish again two cases that follow from $a \trianglelefteq^- b$:
  - $a \trianglelefteq^- b_i$. Then $a \trianglelefteq^- b_i \trianglelefteq^- c_{F(i)}$. By the inductive hypothesis, $a \trianglelefteq^- c_{F(i)}$, thus, by definition, $a \trianglelefteq^- c$.

- There exists an injective $G : I_m \to I_n$ such that for all $i \leq m$, $a_i \trianglelefteq^- b_{G(i)}$. Then for all $i \leq m$, we have $a_i \trianglelefteq^- b_{G(i)} \trianglelefteq^- c_{F(G(i))}$. By the inductive hypothesis, for all $i \leq m$, $a_i \trianglelefteq^- c_{F(G(i))}$. Thus again $a \trianglelefteq^- c$.

$\square$

As a result, from now on $\trianglelefteq$ will be written in stead of $\trianglelefteq^-$. Obviously, $\trianglelefteq$ is a quasi–ordering. By lemma 4.2 $0 \in \mathcal{Q}_{<n}$ is a minimal element.

Lemma 5.3. *The depth function preserves the order —* $s \trianglelefteq t \Rightarrow d(s) \leq d(t)$

Proof. by induction on the complexity of t.

- **Basis** $t = 0$. Then $s \trianglelefteq^- t$ results in $s = 0$, thus $0 = d(s) \leq d(t)$.
- **Inductive** $t = \bullet_n(t_0, \ldots, t_n)$. Also $d(t) = \max\{d(t_0), \ldots, d(t_n)\} + 1$. If $s = 0$, then again $0 = d(s) \leq d(t)$. Otherwise $s = \bullet(s_0, \ldots, s_m)$ and $d(s) = \max\{d(s_0), \ldots, d(s_m)\} + 1$. $s \trianglelefteq^- t$ implies one of the following to hold:
  - $s \trianglelefteq t_i$ for some $i \leq n$. By the inductive hypothesis, $d(s) \leq d(t(i)) \leq \max\{d(t_i)|i \leq n\} + 1 = d(t)$.
  - There exists an injective function $F : I_m \to I_n$ and for all $i \leq n$, $s_i \trianglelefteq^- t_{F(i)}$. By the inductive hypothesis, $d(s_i) \leq d(t_{F(i)})$. Clearly $d(s) = \max\{d(s_i)|i \leq m\} + 1 \leq \max\{d(t_{F(i)})|i \leq m\} + 1 \leq \max\{d(t_i)|i \leq n\} + 1 = d(t)$.

$\square$

Corollary 5.4. *Trivially,* $d(s) > d(t) \Rightarrow s \ntrianglelefteq^- t$.

Definition 5.3. Each $\alpha \leq \omega_n \leq \varepsilon_0$ and thus by the *Cantor Normal Form*, $\alpha = 0$ or $\exists! \alpha_0 \geq \ldots \geq \alpha_n(\alpha = \omega^{\alpha_0} + \ldots + \omega^{\alpha_n})$ with $\alpha_0 < \alpha$. I then define a function $\psi_n : \omega_n \to \mathcal{Q}_{\leq n}$ as follows.

- $\psi_n(0) \doteq 0$.
- $\psi_n(\omega^{\alpha_0} + \ldots + \omega^{\alpha_m}) \doteq \bullet_m (\psi_n(\alpha_0), \ldots, \psi_n(\alpha_m))$.

Lemma 5.5. *For all* $n \in \mathbb{N}$ *and for all* $\alpha \in \omega_n$, $\psi_n(\alpha) \in \mathcal{Q}_{\leq n}$.

Proof. By induction on n. If $n = 0$, then $\alpha \in \omega_0 = 1$, which implies $\alpha = 0$. Thus $\psi_0(\alpha) = 0 \in \mathcal{Q}_{\leq 0}$. If $n > 0$, either $\alpha = 0$, from which follows $\alpha \in \mathcal{Q}_{\leq n}$ for any n, or one finds the Cantor Normal Form expansion and writes $\psi_n(\alpha) = \bullet_m(\psi(\alpha_0), \ldots, \psi_n(\alpha_m))$. It follows that all for all $i < m$, $\alpha_i \in \omega_{n-1}$. By the inductive hypothesis, all $\psi(\alpha_i) \in \mathcal{Q}_{\leq n-1}$ and as a result $\psi(\alpha) \in \mathcal{Q}_{\leq n}$. $\square$

Lemma 5.6. *For any $n \in \mathbb{N}$, for $\mathcal{Q}_{\leq n}$ it holds that $\forall \alpha, \beta(\psi(\alpha) \trianglelefteq \psi(\beta) \Rightarrow \alpha \leq \beta)$*

Proof. By induction on β.

- **Basis** $\beta = 0$. Then $\psi(\beta) = 0$. Since $0 \in \mathcal{Q}_{\leq n}$ is a minimal element, $\psi(\alpha) = 0$. Then $\alpha = 0$. Thus $\alpha \leq \beta$ for all β.

- **Inductive** $\beta > 0$. We assume $\psi(\alpha) \trianglelefteq \psi(\beta)$. If $\psi(\alpha) = 0$ then, as before $\alpha \leq \beta$ and our claim is proven. If $\alpha > 0$ then $\psi(\alpha) \neq 0$. Then we can write $\alpha = \omega^{\alpha_0} + \ldots + \omega^{\alpha_r}$ and $\psi(\alpha) = \bullet_r(\psi(\alpha_0), \ldots, \psi(\alpha_r))$ and likewise $\beta = \omega^{\beta_0} + \ldots + \omega^{\beta_p}$ and $\psi(\beta) = \bullet_p(\psi(\beta_0), \ldots, \psi(\beta_p))$. By definition of $\trianglelefteq$ one of the following is the case:
  - $\exists i(\psi(\alpha) \trianglelefteq \psi(\beta_i))$. By inductive hypothesis, $\alpha \leq \beta_i$. Then, by lemma 2.1, $\alpha \leq \beta$.
  - $\exists F : I_r \to I_p(F(i) = F(j) \Rightarrow i = j \wedge \psi(\alpha_i) \trianglelefteq \psi(\beta_{F(i)}))$. By inductive hypothesis $\alpha_i \leq \beta_{F(i)}$. Again, by lemma 2.1, $\omega^{\alpha_i} \leq \omega^{\beta_{F(i)}}$. Thus it follows that $\alpha = \omega^{\alpha_0} \# \ldots \# \omega^{\alpha_r} \leq \omega^{\beta_{F(0)}} \# \ldots \# \omega^{\beta_{F(r)}} \leq \omega^{\beta_0} \# \ldots \# \omega^{\beta_p} = \beta$.

$\square$

Definition 5.4. For all $n \in \mathbb{N}$, $\bullet_0^n \in \mathcal{T}$ is defined as follows

$$\begin{aligned} \bullet_0^0 &= 0 \\ \forall n \in \mathbb{N}, n > 0 \quad \bullet_0^n &= \bullet_0(\bullet_0^{n-1}) \end{aligned}$$

It is clear that $\bullet_0^n \in \mathcal{Q}_n$.

Definition 5.5. For all $n \in \mathbb{N}$ there is defined $\pi_n : \mathcal{Q}_{\leq n} \to \mathcal{T}$ as follows:

$$\begin{aligned} \pi_n(0) &= \bullet_0^n \\ \forall m \in \mathbb{N} \quad \pi_n(\bullet_m(t_0, \ldots, t_m)) &= \bullet_{m+1}(\bullet_0^{n-1}, t_0, \ldots, t_m) \end{aligned}$$

Lemma 5.7. $\pi[\mathcal{Q}_{\leq n}] \subset \mathcal{Q}_n$

Proof. One considers any $t \in \mathcal{Q}_{\leq n}$. If $t = 0$, then $\pi(t) = \bullet_0^n$ and $d(\pi(t)) = n$. If $t \neq 0$ then, for some m, $t = \bullet_m(t_0, \ldots, t_m)$ with, for all $i \leq m$, $t_i \in \mathcal{Q}_{\leq n-1}$ and so $d(t_i) \leq n - 1$ and $n \geq d(t) = \max\{d(t_0), \ldots, d(t_m)\} + 1$. Then $\pi(t) = \bullet_{m+1}(\bullet_0^{n-1}, t_0, \ldots, t_m)$. This implies that $d(\pi(t)) = \max\{n - 1, d(t_0), \ldots, d(t_m)\} + 1$. Thus $d(\pi(t)) \geq n$. Also $d(\pi(t)) \leq n$, since $d(t_i) \leq n$. □

Lemma 5.8. *For all $n \in \mathbb{N}$ and $s \in \mathcal{T}$ it holds that $s \trianglelefteq \bullet_0^n \Rightarrow \exists m \leq n(s = \bullet_0^m)$.*

Proof. Induction on n.

- **Basis** If $n = 0$, then $\bullet_0^n = 0$. And $s \trianglelefteq 0 \Rightarrow s = 0$, and $0 \leq n$.
- **Inductive** $n > 0$. If $s = 0$ then the result follows as above. Otherwise, for some $r \in \mathbb{N}$, $s = \bullet_r(s_0, \ldots, s_r)$. Consequently $s \trianglelefteq \bullet_0^n = \bullet(\bullet_0^{n-1})$ implies that either (1) $s \trianglelefteq \bullet_0^{n-1}$, which by the inductive hypothesis implies that $s = \bullet_0^m$ for some $m \leq n - 1 \leq n$, thus the claim is proven, or (2) there exists an injective function $F : I_r \to I_0$, which means that $r = 0$ and thus $s_0 \trianglelefteq \bullet_0^{n-1}$. By the inductive hypothesis there exists $m' \in \mathbb{N}$ such that $s_0 = \bullet_0^{m'}$ and $s = \bullet_0^{m'+1}$.

□

Lemma 5.9. *For any $a \in \mathcal{T}$, $n > 0$, $\bullet_0^n \trianglelefteq a \Rightarrow \bullet_0^{n-1} \trianglelefteq a$*

Proof. Clearly, for $n > 0$, $\bullet_0^{n-1} \trianglelefteq \bullet_0^n = \bullet_0(\bullet^{n-1})$, since $\bullet^{n-1} \trianglelefteq \bullet^{n-1}$. Thus, since $\trianglelefteq$ is transitive, $\bullet_0^{n-1} \trianglelefteq a$. □

Lemma 5.10. *For all $n \in \mathbb{N}$ it holds that $\pi_n(s) \trianglelefteq \pi_n(t) \Rightarrow s \trianglelefteq t$.*

Proof. By induction on the complexity of $t \in \mathcal{Q}_{\leq n}$.

- **Basis** If $t = 0$ then $\pi(t) = \bullet_0^n$, and, by lemma 5.8, for some $m \in \mathbb{N}$, $s = \bullet_0^m$. By lemma 5.7 it must be that $m = n$. Thus $s = 0$ and $s \trianglelefteq t$.
- **Inductive** If $s = 0$ then the argument holds as before. Otherwise $t = \bullet_p(t_0, \ldots, t_p)$ and $s = \bullet_r(s_0, \ldots, s_r)$. Then $\pi(t) = \bullet_{p+1}(\bullet_0^{n-1}, t_0, \ldots, t_p)$ and $\pi(s) = \bullet_{r+1}(\bullet_0^{n-1}, s_0, \ldots, s_r)$. From $\pi(s) \trianglelefteq \pi(t)$ it follows that either one of the following holds:

– $\pi(s) \trianglelefteq \bullet_0^{n-1}$. Thus $\pi(s) = \bullet_0^m$ for some $m < n$, but then $\pi(s) \notin \mathcal{Q}_n$. Absurd.
– There exists $i \leq p$, $\pi(s) \trianglelefteq t_i$. Absurd, since $d(\pi(s)) > d(t_i)$.
– There exists an injective function $F : I_{r+1} \to I_{p+1}$. If $0 \notin F[I_p]$ then $G : I_p \to I_r$ is defined by $G(i) = F(i+1)$ and injective and thus $s \trianglelefteq t$. If $0 \in F[I_p]$ then $\exists q \leq p(F(q) = 0)$ and $s_q \trianglelefteq \bullet_0^{n-1}$. By lemma 5.8 for some $m < n$, $s_{F(q)} = \bullet_0^m$. Furthermore, $\bullet_0^{n-1} \trianglelefteq t_{F(0)}$. By lemma 5.9, $s_q \trianglelefteq t_{F(0)}$. Thus the function $H : I_p - \{0\} \to I_r$ defined such that $H(i) = F(i)$ for all $i \neq q$ and $H(q) = F(0)$ is still injective and thus, as above, $s \trianglelefteq t$.

□

Lemma 5.11. *For any $n \in \mathbb{N}$, there exists a function $\psi' : \omega_n \to \mathcal{Q}_n$ such that it holds in $\mathcal{Q}_n$ that $\forall \alpha, \beta(\psi'(\alpha) \trianglelefteq \psi'(\beta) \Rightarrow \alpha \leq \beta)$.*

Proof. We take any $n \in \mathbb{N}$. By lemma 5.6 there exists a function $\psi_n : \omega_n \to \mathcal{Q}_{\leq n}$ such that $\forall \alpha, \beta \in \omega_n(\psi_n(\alpha) \trianglelefteq \psi_n(\beta) \Rightarrow \alpha \leq \beta$. By lemma 5.10 there exists a function $\pi : \mathcal{Q}_{\leq n} \to \mathcal{Q}_n$ such that $\forall s, t \in \mathcal{Q}_{\leq n}(\pi_n(s) \trianglelefteq \pi_n(t) \Rightarrow s \trianglelefteq t)$.

Then one defines $\psi'_n = \pi_n \circ \psi_n$. Clearly if $\psi'_n(\alpha) \trianglelefteq \psi'_n(\beta)$, $\pi_n(\psi_n(\alpha)) \trianglelefteq \pi_n(\psi_n(\beta))$, and thus $\psi_n(\alpha) \trianglelefteq \psi_n(\beta)$ and finally $\alpha \leq \beta$.

The argument can be repeated to yield the same result for any $n \in \mathbb{N}$. □

Lemma 5.12. $\mathrm{ACA}_0 \vdash \forall n \mathrm{CWF}(\omega_n)$ *implies that* $\mathrm{ACA}_0 \vdash \mathrm{CWF}(\varepsilon_0)$

Proof. $\mathrm{ACA}_0 \vdash \mathrm{CWF}(\varepsilon_0)$ is equivalent to $\mathrm{ACA}_0 \vdash \forall F : \mathbb{N} \to A \exists i, j[i < j \wedge F(i) \leq F(j)]$. Assume the denial of the consequent, thus $\mathrm{ACA}_0 \vdash \exists F : \mathbb{N} \to \varepsilon_0 [\forall i, j[i < j \Rightarrow F(i) \nleq F(j)]]$. Take such an F. We argue that $\exists n F[\mathbb{N}] \subset \omega_n$, since (1) the ordering $(\varepsilon_0, \leq)$ is total and hence $F(i) \nleq F(j)$ implies $F(i) > F(j)$, and (2) from $F(0) \in \varepsilon_0$ one deduces that $F(0) \leq \omega_n$ for some $n \in \omega$.

Thus more precisely $F : \mathbb{N} \to \omega_n$ and F is, by assumption, such that $\forall i, j[i < j \Rightarrow F(i) \nleq F(j)]$ and thus $\mathrm{ACA}_0 \nvdash \forall n \mathrm{CWF}(\omega_n)$. □

Theorem 5.13. $\mathrm{ACA}_0 \nvdash \forall n(\mathrm{WQ}(\mathcal{Q}_n))$.

Proof. It is assumed to be known that

$$\mathrm{ACA}_0 \nvdash \mathrm{CWF}(\varepsilon_0) \Rightarrow \mathrm{ACA}_0 \nvdash \forall F : \mathbb{N} \to \varepsilon_0, \exists i, j(i < j \wedge F(i) \leq F(j)) \quad (9)$$

Proof by absurdity. Let us assume that

$$\mathrm{ACA}_0 \vdash \forall n(\mathrm{WQ}(\mathcal{Q}_n)) \tag{10}$$

Let us take, for any $n \in \mathbb{N}$, any function $G : \mathbb{N} \to \omega_n$.

By lemma 5.11 there exists at least a function $\psi : \omega_n \to \mathcal{Q}_n$ such that $\psi(\alpha) \trianglelefteq \psi(\beta) \Rightarrow \alpha \leq \beta$. Now $F \doteq \psi \circ G$. Thus $F : \mathbb{N} \to \mathcal{Q}_n$. By our assumption 8 it must be that $\exists i, j(i < j \wedge F(i) \leq F(j))$. This means that $F(i) = \psi(G(i)) \leq \psi(G(j)) = F(j)$. Then $G(i) \leq G(j)$. Due to the arbitrariness of G, it must be that $\mathrm{ACA}_0 \vdash \mathrm{CWF}(\omega_n)$. Due to the arbitrariness of n, $\mathrm{ACA}_0 \vdash \forall n(\mathrm{CWF}(\omega_n))$. By lemma 5.12, we would have $\mathrm{ACA}_0 \vdash \mathrm{CWF}(\varepsilon_0)$, which is absurd. □

6 Proposition iii

Definition 6.1. The set of all exactly binary trees $\mathcal{B}_2 \subset \mathcal{T}$ will be defined as the smallest set $\mathcal{B}_2$ such that[1]

$$0 \in \mathcal{B}_2$$
$$t_1, t_2 \in \mathcal{B}_2 \Rightarrow \bullet(t_1, t_2) \in \mathcal{B}_2$$

Definition 6.2. Then $\sim$ is the smallest possible relation such that

$$0 \sim 0$$
$$\forall t_0, t_1, s_0, s_1 \in \mathcal{B}_2 \qquad \bullet(t_0, t_1) \sim \bullet(s_0, s_1) \Leftrightarrow \begin{array}{l}(t_0 \sim s_0 \wedge t_1 \sim s_1) \vee \\ (t_1 \sim s_0 \wedge t_0 \sim s_1)\end{array}$$

Lemma 6.1. *$\sim$ is an equivalence relation.*

Proof. The relation has the required properties:

Reflexive By induction, $s = t \Rightarrow s \sim t$.

Symmetric Let $s \sim t$. If $s = 0$ or $t = 0$ then $s = t = 0$, otherwise there exists a smaller relation $\sim$ with the above conditions. Let $s = \bullet(s_0, s_1)$ and $t = \bullet(t_0, t_1)$. It follows immediately from the definition of $\sim$ that $s \sim t \Leftrightarrow t \sim s$. Thus $t \sim s$.

[1] The subscript will be dropped for $\bullet$ such that $\bullet(a_0, a_1)$ is taken to represent $\bullet_1(a_0, a_1)$.

Transitive Induction on the complexity of u. Let $s \sim t$ and $t \sim u$. As above, if $u = 0$ then $t = 0$, therefore $s = 0$, thus $s = t = u = 0$ and by the reflexivity $s \sim u$. Otherwise $s = \bullet(s_0, s_1) \sim \bullet(t_0, t_1) = t$ and $t = \bullet(t_0, t_1) \sim \bullet(u_0, u_1) = u$. This means $\exists i, j \in \{0, 1\}$ such that $s_i \sim t_0$ and $s_{1-i} \sim t_1$ and $t_j \sim u_0$ and $t_{1-j} \sim u_1$. By inductive hypothesis $s_0 \sim u_0$ and $s_1 \sim u_1$ or $s_0 \sim u_1$ and $s_1 \sim u_0$. Thus $s \sim u$.

□

Definition 6.3. Then there is defined $\trianglelefteq$ as the smallest possible relation in $\mathcal{B}_2$ such that

$$\begin{array}{lll} \forall t \in \mathcal{B}_2 & 0 \trianglelefteq t & \\ \forall s_1, s_2, t_1, t_2 \in \mathcal{B}_2 & \bullet(s_1, s_2) \trianglelefteq \bullet(t_1, t_2) \Leftrightarrow & (\bullet(s_1, s_2) \trianglelefteq t_1) \vee \\ & & (\bullet(s_1, s_2) \trianglelefteq t_2) \vee \\ & & (s_1 \trianglelefteq t_1 \wedge s_2 \trianglelefteq t_2) \vee \\ & & (s_2 \trianglelefteq t_1 \wedge s_1 \trianglelefteq t_2) \end{array}$$

Lemma 6.2. *If $a' \sim a \trianglelefteq b \sim b'$, then $a' \trianglelefteq b'$.*

Proof. By induction on the complexity of $b \in \mathcal{B}_2$.

- **Basis** $b = 0$. Then $a = 0$ and thus $a' = 0$, so that $a' \trianglelefteq b'$ for any $b' \in \mathcal{B}_2$.

- **Inductive** $b = \bullet(b_0, b_1)$. Then $0 \neq b' = \bullet(b'_0, b'_1)$. From $b \sim b'$ one deduces that $b_0 \sim b'_0 \wedge b_1 \sim b'_1$ or $b_0 \sim b'_1 \wedge b_1 \sim b'_0$. If $a = 0$, then, as before, $a' \trianglelefteq b'$ for any $b' \in \mathcal{B}_2$. Otherwise $a = \bullet(a_0, a_1)$. Since $a \trianglelefteq b$ one of the following holds:

  1. $a \trianglelefteq b_0$. If $b_0 \sim b'_0$ then, by the inductive hypothesis, $a \trianglelefteq b'_0$. Thus $a' \sim a \trianglelefteq b'$, therefore $a' \trianglelefteq b'$. If $b_0 \sim b'_1$, then $a' \sim a \trianglelefteq b'_1$ and $a' \trianglelefteq b'$.
  2. $a \trianglelefteq b_1$. As before.
  3. $a_0 \trianglelefteq b_0 \wedge a_1 \trianglelefteq b_1$. From $a \sim a'$ we deduce $a_0 \sim a'_0 \wedge a_1 \sim a'_1$ or $a_0 \sim a'_1 \wedge a_1 \sim a'_0$. Let us assume the first case holds, but the proof of the other is entirely symmetrical. If $b_0 \sim b'_0$ and $b_1 \sim b'_1$ then $a'_0 \sim a_0 \trianglelefteq b_0 \sim b'_0$ and $a'_1 \sim a_1 \trianglelefteq b_1 \sim b'_1$. Hence by the inductive hypothesis $a'_0 \trianglelefteq b'_0$ and $a'_1 \trianglelefteq b'_1$. Thus $a' \trianglelefteq b'$. If $b_0 \sim b'_1$ and $b_1 \sim b'_0$ then, by the inductive hypothesis, $a'_0 \trianglelefteq b'_1$ and $a'_1 \trianglelefteq b'_0$, from which again $a' \trianglelefteq b'$.

4. $a_0 \trianglelefteq b_1 \wedge a_1 \trianglelefteq b_0$. As before.

□

Corollary 6.3. *If* $s \sim t$ *then* $s \trianglelefteq t$

Proof. $t \trianglelefteq t$ since $\trianglelefteq$ is reflexive. Together with the assumption $s \sim t$ this leads to $s \trianglelefteq t$. □

In particular, $\trianglelefteq$ is reflexive. Also, it is transitive, since lemma 5.2 will produce the same result now that the only juxtaposing function is the binary $\bullet_1$. Thus, $\trianglelefteq$ is a quasi–ordering.

Lemma 6.4. $a \sim a' \Rightarrow d(a) = d(a')$

Proof. From $a \sim a'$ follows $a \trianglelefteq a'$, and thus by lemma 5.3, $d(a) \leq d(a')$. Additionally, $a' \trianglelefteq a$ and therefore, as before, $d(a') \leq d(a)$ whence $d(a) = d(a')$. □

By *Cantor's Normal Form*,

$$\forall \alpha \neq 0, \alpha < \varepsilon_0 (\exists! \alpha_1 \geq \ldots \geq \alpha_n (\alpha = (\omega^{\alpha_1} + \ldots + \omega^{\alpha_{n-1}}) + \omega^{\alpha_n}))) \qquad (11)$$

In particular, for $\alpha \neq 0, \alpha < \varepsilon_0$, $\exists \beta < \alpha, \exists \gamma < \alpha(\alpha = \beta + \omega^\gamma)$. We define a function $\phi : \mathrm{On} \times \mathrm{On} \to \mathrm{On}$ as $\phi(\alpha, \beta) = \alpha + \omega^\beta$.

Definition 6.4. For each $A \in \mathcal{B}_2$ and $m \in \omega$ there is defined by induction:

$$\begin{aligned} \odot_0^A &= A \\ \odot_m^A &= \bullet(A, \odot_{m-1}^A) \end{aligned}$$

Clearly $A \trianglelefteq \odot_n^A$ for any $n \in \omega$.

Definition 6.5. For each $A, B \in \mathcal{B}$ we have

$$\begin{aligned} A(B) &= B && \text{If } A = 0 \\ A(B) &= \bullet(A_1, A_2(B)) && \text{If } A = \bullet(A_1, A_2) \end{aligned}$$

Clearly $A = A(0) = 0(A)$. Also clearly $A \trianglelefteq B \Rightarrow A \trianglelefteq B(C)$, for any $C \in \mathcal{T}$.

Definition 6.6. For each $0 < \alpha < \varepsilon_0$ we write the Cantor Normal Form $\alpha = \omega^{a_0} \cdot m_0 + (\ldots + \omega^{a_n} \cdot m_n)$, with $m_i < \omega$ and $a_0 \geq \ldots \geq a_n$. One defines $\alpha_0 = a_0$ and $\alpha_1 = \omega^{a_1} + \ldots + \omega^{a_n}$. Then clearly $\alpha = \omega^{\alpha_0} \cdot m + \alpha_1$. We then define a function $\psi : \varepsilon_0 \to \mathcal{B}_2$ by induction on the argument:

$$\begin{aligned} \psi(0) &= 0 \\ \psi(\omega^{\alpha_0} \cdot m + \alpha_1) &= \odot_m^{\psi(\alpha_0)}(\psi(\alpha_1)) \end{aligned}$$

Lemma 6.5. $\bullet(A, 0) \ntrianglelefteq A$.

Proof. By induction on the complexity of $A \in \mathcal{B}_2$.

1. $A = 0$. Clearly $\bullet(0, 0) \ntrianglelefteq 0$.
2. $A = \bullet(A_1, A_2)$. If $\bullet(A, 0) \trianglelefteq A$ then one of the following holds
   (a) $\bullet(A_1, A_2) \trianglelefteq A_1$. But then $\bullet(A_1, 0) \trianglelefteq A_1$, which, by induction, does not hold.
   (b) $\bullet(A_1, A_2) \trianglelefteq A_2$. As above.
   (c) $\bullet(A, 0) \trianglelefteq A_1$. But then $\bullet(A_1, 0) \trianglelefteq \bullet(\bullet(A_1, 0), 0) \trianglelefteq \bullet(\bullet(A_1, A_2)) \trianglelefteq A_1$, contrary to the inductive hypothesis.
   (d) $\bullet(A, 0) \trianglelefteq A_2$. Then $\bullet(A_2, 0) \sim \bullet(0, A_2) \trianglelefteq \bullet(A_2, 0) \trianglelefteq A_2$, contrary to the inductive hypothesis.

□

Corollary 6.6. $\bullet(A, B) \ntrianglelefteq A$

Proof. $\bullet(A, 0) \trianglelefteq \bullet(A, B) \trianglelefteq A$ is absurd. □

Corollary 6.7. $A = \bullet(A_1, A_2) \Rightarrow A \ntrianglelefteq A_1$.

Lemma 6.8. $A \trianglelefteq A' \trianglelefteq A \Rightarrow A \sim A'$

Proof. Induction on the complexity of A.

- **Basis** $A = 0$. Then from the assumption it follows that $A' = 0$. Thus $A \sim A'$.

- **Inductive** $A = \bullet(A_1, A_2)$. Then from the assumption $A \trianglelefteq A'$ it follows that $A' \neq 0$. Let $A' = \bullet(A'_1, A'_2)$. First the claim is $A' \ntrianglelefteq A_i$, for $i \in \{1, 2\}$. This is obvious from $A \trianglelefteq A' \trianglelefteq A_1$, which, by corollary 6.7 is absurd. Likewise $A \ntrianglelefteq A'_i$. Then from the assumption $A' \trianglelefteq A \wedge A \trianglelefteq A'$ it follows that:

  1. $A'_1 \trianglelefteq A_1 \wedge A'_2 \trianglelefteq A_2$.
     (a) $A_1 \trianglelefteq A'_1 \wedge A_2 \trianglelefteq A'_2$. Then $A_1 \trianglelefteq A'_1 \trianglelefteq A_2 \trianglelefteq A'_2$. Thus, by the inductive hypothesis, $A_1 \sim A'_1$ and $A_2 \sim A'_2$. As a consequence, $A \sim A'$.
     (b) $A_2 \trianglelefteq A'_1 \wedge A_1 \trianglelefteq A'_2$. Then $A_1 \trianglelefteq A'_1 \trianglelefteq A'_2 \trianglelefteq A_2 \trianglelefteq A'_1 \trianglelefteq A_1$, thus $A_1 \sim A'_1 \sim A'_2 \sim A_2$. As a result, $A \sim A'$.
  2. $A'_2 \trianglelefteq A_1 \wedge A'_1 \trianglelefteq A_2$. Similarly.

□

Corollary 6.9. *If $A' \trianglelefteq A$ but not $A \sim A'$ then $A \ntrianglelefteq A'$.*

Lemma 6.10. $\bullet(A, B) \trianglelefteq \bullet(A, C) \Rightarrow B \trianglelefteq C$.

Proof. The assumption implies that one of the following holds:

1. $\bullet(A, B) \trianglelefteq A$. Absurd according to corollary 6.7.
2. $\bullet(A, B) \trianglelefteq C$. Then $B \trianglelefteq \bullet(A, B) \trianglelefteq C$.
3. $A \trianglelefteq A \wedge B \trianglelefteq C$. Clear.
4. $A \trianglelefteq C \wedge B \trianglelefteq A$. Then $B \trianglelefteq C$.

□

Corollary 6.11. $A(B) \trianglelefteq A(C) \Rightarrow B \trianglelefteq C$

Proof. Induction on the complexity of A.

- **Basis** If $A = 0$ then $B = A(B) \trianglelefteq A(C) = C$
- **Inductive** If $A = \bullet(A_1, A_2)$ then we have $\bullet(A_1, A_2(B)) \trianglelefteq \bullet(A_1, A_2(C))$ and thus $A_2(B) \trianglelefteq A_2(C)$. By the inductive hypothesis, $B \trianglelefteq C$.

□

Definition 6.7. For all $A, B \in \mathcal{T}$, $A \lhd B \Leftrightarrow \exists B_1, B_2(B = \bullet(B_1, B_2) \wedge (A \unlhd B_1 \vee A \unlhd B_2))$.

Lemma 6.12. $\lhd$ *has the following properties:*

1. $A \not\lhd A$.
2. $A \lhd B \lhd C \Rightarrow A \lhd C$.
3. $A \unlhd B \lhd C \Rightarrow A \lhd C$.
4. $A \lhd B \Rightarrow A \unlhd B \wedge A \not\sim B$.

Proof. As follows:

1. By corollary 6.7, if $A = \bullet(A_1, A_2)$ then not $A \unlhd A_1$, nor $A \unlhd A_2$ (since then $\bullet(A_2, A_1) \sim A \unlhd A_2$. Consequently $A \not\lhd A$.
2. From $A \lhd B \lhd C$ follows that $A \unlhd B_i \unlhd B \unlhd C_j$, hence $A \unlhd C_j$ and finally $A \lhd C$.
3. Clearly $A \unlhd B \unlhd C_i$ implies $A \unlhd C_i$ and thus $A \lhd C$.
4. From $A \unlhd B$ we deduce $A \unlhd B_i$, whence $A \unlhd B$. If we would have $A \lhd B$ and $A \sim B$ then also $B \unlhd A \lhd B$, whence, by the previous result, $B \lhd B$, absurd.

□

Lemma 6.13. $\bullet(A, B) \unlhd C(D) \Rightarrow A \lhd C \vee B \lhd C \vee \bullet(A, B) \unlhd D$.

Proof. By induction on the complexity of C.

- **Basis** $C = 0$. Then $\bullet(A, B) \unlhd D$, which implies the consequent.
- **Inductive** $C = \bullet(C_1, C_2)$. Then $\bullet(A, B) \unlhd \bullet(C_1, C_2(D))$ implies that one of the following holds:
    1. $\bullet(A, B) \unlhd C_1$. Then $A \unlhd \bullet(A, B) \unlhd C_1 \lhd C$.
    2. $\bullet(A, B) \unlhd C_2(D)$. By the inductive hypothesis, $A \lhd C_2$, whence $A \lhd C$, or $B \lhd C_2$, whence $B \lhd C$, or $\bullet(A, B) \unlhd D$.
    3. $A \unlhd C_1 \wedge B \unlhd C_2(D)$, then $A \lhd C$.

4. $A \trianglelefteq C_2(D) \wedge B \trianglelefteq C_1$, then $B \lhd C$.

$\square$

Corollary 6.14. $\bullet(A, A)(B) \trianglelefteq A(C) \Rightarrow \bullet(A, A)(B) \trianglelefteq C$.

Proof. $\bullet(A, A)(B) = \bullet(A, A(B))$. The hypothesis implies one of the following to hold:

1. $A \lhd A$. Absurd, by lemma 6.7.

2. $A(B) \lhd A$. Equally $A \trianglelefteq A(B) \lhd A$. Absurd.

3. $\bullet(A, A)(B) = \bullet(A, A(B)) \trianglelefteq C$.

$\square$

Corollary 6.15. *For $q > 0$, $\odot_q^A(B) \trianglelefteq A(C) \Rightarrow \odot_q^A(B) \trianglelefteq C$.*

Proof. For $q = 1$, this is the lemma. For $q > 1$, $\odot_q^A(B) = \bullet(A, \odot_{q-1}^A(B)) \trianglelefteq A(C)$ lemma 6.13 implies that one of the following holds:

1. $A \lhd A$. Absurd by lemma 6.7.

2. $\odot_{q-1}^A(B) \lhd A$. But then $A \trianglelefteq \odot_{q-1}^A(B) \lhd A$. Absurd.

3. $\odot_q^A(B) \trianglelefteq C$, which is the only remaining option.

$\square$

Lemma 6.16. *For all $m > n$, $\odot_m^A(B) \trianglelefteq \odot_n^A(C) \Rightarrow \odot_{m-n}^A(B) \trianglelefteq \odot_0^A(C)$.*

Proof. By induction on n.

- **Basis**. $n = 0$. Then it is implied that $\odot_{m-0}^A(B) \trianglelefteq \odot_0^A(C)$ from the hypothesis.

- **Inductive** $n > 0$. Then also $m > 0$. One can write $\bullet(A, \odot_{m-1}^A)(B) = \bullet(A, \odot_{m-1}^A(B)) \trianglelefteq \odot_n^A(C) = \bullet(A, \odot_{n-1}^A(C))$. This implies, according to lemma 6.10, $\odot_{m-1}^A(B) \trianglelefteq \odot_{n-1}^A(C)$, from which, by the inductive hypothesis, one concludes $\odot_{m-n}^A(B) \trianglelefteq \odot_0^A(C)$.

$\square$

Lemma 6.17. *For $m > n$, $\odot_m^A(B) \trianglelefteq \odot_n^A(C) \Rightarrow \odot_{m-n}^A(B) \trianglelefteq C$.*

Proof. By lemma 6.16 it follows that $\odot_{m-n}^A(B) \trianglelefteq \odot_0^A(C) = A(C)$. By corollary 6.15 $\odot_{m-n}^A(B) \trianglelefteq C$. $\square$

Lemma 6.18. $\psi(\alpha) \trianglelefteq \psi(\beta) \Rightarrow \alpha \leq \beta$

Proof. Induction on β. If $\beta = 0$ then $\psi(\beta) = 0$, and $\psi(\alpha) = 0$, from which $\alpha = 0 \leq \beta$. Otherwise $\beta = \omega^{\beta_0} \cdot n + \beta_1$. If $\alpha = 0$ clearly $\alpha \leq \beta$. Let us assume that $\alpha = \omega^{\alpha_0} \cdot m + \alpha_1$. Then $\psi(\alpha) = \odot_m^{\psi(\alpha_0)}(\psi(\alpha_1))$ and $\psi(\beta) = \odot_n^{\psi(\beta_0)}(\psi(\beta_1)$.

Clearly $m > 0$. Induction on n.

- **Basis** The case $n = 0$ will not appear unless referred to from the inductive clause. $\psi(\alpha) = \bullet(\psi(\alpha_0), \odot_{m-1}^{\psi(\alpha_0)}(\psi(\alpha_1))) \trianglelefteq \psi(\beta_0)(\psi(\beta_1)) = \odot_{n-1}^{\psi(\beta_0)}(\psi(\beta_1))$, by lemma 6.13, this implies that either (1) $\psi(\alpha) \trianglelefteq \psi(\beta_1)$, whence $\alpha \leq \beta_1 \leq \beta$, or otherwise at least (2) $\psi(\alpha_0) \trianglelefteq \psi(\alpha) \triangleleft \psi(\beta_0)$, whence $\alpha_0 \leq \beta_0$. If $\alpha_0 < \beta_0$ then also $\alpha \leq \beta$. If $\alpha_0 = \beta_0$ then $\psi(\beta_0) = \psi(\alpha_0) \triangleleft \psi(\beta_0)$, absurd.

- **Inductive** $\psi(\alpha) = \odot_m^{\psi(\alpha_0)}(\psi(\alpha_1))$ and $\psi(\beta) = \odot_n^{\psi(\beta_0)}(\psi(\beta_1))$. $\psi(\alpha) \leq \psi(\beta)$ implies that one of the following holds:

  1. $\psi(\alpha) \trianglelefteq \psi(\beta_0)$. By the inductive hypothesis $\alpha \leq \beta_0 \leq \beta$
  2. $\psi(\alpha) \trianglelefteq \odot_{n-1}^{\psi(\beta_0)}(\psi(\beta_1))$. By the inductive hypothesis of the induction on n it follows that $\alpha \leq \omega^{\beta_0} \cdot (n-1) + \beta_1 \leq \beta$.
  3. $\psi(\alpha_0) \trianglelefteq \psi(\beta_0)$ and $\odot_{m-1}^{\psi(\alpha_0)}(\psi(\alpha_1)) \trianglelefteq \odot_{n-1}^{\psi(\beta_0)}(\psi(\beta_1))$. From the former fact it follows, by the inductive hypothesis, that $\alpha_0 \leq \beta_0$. If $\alpha_0 < \beta_0$ then $\alpha \leq \beta$. If, however, $\alpha_0 = \beta_0$, then let us consider m and n.
     (a) If $m < n$ then also $\alpha \leq \beta$.
     (b) If $m = n$ then $\odot_{m-1}^{\psi(\alpha_0)}(\psi(\alpha_1)) \trianglelefteq \odot_{m-1}^{\psi(\beta_0)}(\psi(\beta_1))$, and, following corollary 6.11, $\psi(\alpha_1) \leq \psi(\beta_1)$, whence $\alpha = \omega^{\alpha_0} \cdot m + \alpha_1 \leq \omega^{\beta_0} \cdot m + \beta_1 = \beta$.
     (c) If $m > n$ then $\odot_{m-1}^{\psi(\alpha_0)}(\psi(\alpha_1)) \trianglelefteq \odot_{n-1}^{\psi(\alpha_0)}(\psi(\beta_1))$. By lemma 6.17 it follows that $\psi(\omega^{\alpha_0} \cdot (m-n) + \alpha_1) = \odot_{m-n}^{\psi(\alpha_0)} \trianglelefteq \psi(\beta_1)$. By the inductive hypothesis, $\omega^{\alpha_0} \cdot (m-n) + \alpha_1 \leq \beta_1$. Consequently $\alpha = \omega^{\alpha_0} \cdot m + \alpha_1 \leq \omega^{\alpha_0} \cdot n + \beta_1 = \beta$.

4. $\odot_{m-1}^{\psi(\alpha_0)}(\psi(\alpha_1)) \trianglelefteq \psi(\beta_0)$ and $\psi(\alpha_0) \trianglelefteq \odot_{n-1}^{\psi(\beta_0)}(\psi(\beta_1))$. Then in particular $\psi(\alpha_0) \trianglelefteq \psi(\beta_0)$. From the former assumption it follows moreover that $\odot_{m-1}^{\psi(\alpha_0)}(\psi(\alpha_1)) \trianglelefteq \psi(\beta_0) \trianglelefteq \odot_{n-1}^{\psi(\beta_0)}(\psi(\beta_1))$. Thus the previous argument can be repeated.

□

Corollary 6.19. $\psi(\alpha) \sim \psi(\beta) \Rightarrow \alpha = \beta$

Proof. As before, $\psi(\alpha) \trianglelefteq \psi(\beta)$ and thus $\alpha \leq \beta$. Likewise $\beta \leq \alpha$ and thus $\alpha = \beta$. □

Theorem 6.20. $\mathrm{ACA}_0 \nvdash \mathrm{WQ}(\mathcal{B}_2)$

Proof. Theorem 4.13. The assumption is lemma 6.18. □

7 Proposition i

The set of all binary trees contains the set of all exactly binary trees. Thus $\mathcal{B}_2 \subset \mathcal{B}$.

Theorem 7.1. $\mathrm{ACA}_0 \nvdash \mathrm{WQ}(\mathcal{B})$

Proof. Theorem 4.13. The function given in 6.18 is $\psi : \varepsilon_0 \to \mathcal{B}_2$, thus, in particular, $\psi : \varepsilon_0 \to \mathcal{B}$. □

8 Conclusion

In conclusion, the explicit proofs for the four propositions set out in the introduction have been given. The result helps to illuminate the correspondence between the ordinal numbers below ε_0 and the trees in such a way that what is known about the structure of the former can be extrapolated to conclusions about the latter.

The propositions are also examples of relevant and meaningful mathematical sentences that are independent of axiomatic systems.

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