On the Parametrisation of Regular Elementary Mathematical Shapes in Three-Dimensional Space

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Abstract

In this paper Geometric Algebra will be applied to the problem of finding a parametrisation of elementary mathematical shapes in three-dimensional space. The examples of a sphere and a single torus will be used. It will be shown that these parametrisations can be given in both exponential and trigonometrical notation and that they are equivalent. Secondly, using the parametrisation the tangent vectors along the surface will be calculated by differentiating the parametrisation over the parameters. Finally the surface area of the shapes will be calculated by integrating the magnitude of the wedge product over the domains of the parameters.

1 Introduction

1.1 Outline

The sphere and the torus will be considered elementary mathematical shapes, and for each a parametrisation in three-dimensional space will be found, and consequently an expression for their tangent vectors, and finally, with the help of this, their surface area and volume.

In both cases choose the simplest coordinate system will be chosen, since once there is an expression for the parametrisation of them in any coordinate system, it can be applied in any coordinate system via simple rotation and reflection – which, especially in Geometric Algebra, has become easy. The free choice of the coordinate system is the result of the rotation and translation invariance of the problem.

1.2 Notation

In this paper, bivectors will be notated using the wedge, such that, for vectors a, b, c:

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c \qquad \text{and} \qquad a \wedge b = -b \wedge a \tag{1}$$

Also, the geometric product will be assumed to be defined as the sum of the inner (dot-) and outer (wedge-)product:

$$ab = a \cdot b + a \wedge b \tag{2}$$

And

$$a \cdot b = \frac{1}{2}(ab + ba) \qquad a \wedge b = \frac{1}{2}(ab - ba) \tag{3}$$

Orthonormal vectors will be used and denoted as e_n, with e_0 corresponding to the time-dimension. Therefore the following, by definition, holds (note that, due to their obvious vector nature, any arrows or bold face in these variables will, for convenience of notation and reading, be omitted):

$$\begin{aligned} e_i \cdot e_j &= 0 \qquad \text{for } i \neq j \\ e_i \cdot e_i &\equiv 1 \qquad \text{for } i = 0 \\ e_i \cdot e_i &\equiv -1 \qquad \text{for } i > 0 \end{aligned} \tag{4}$$

2 Parametrising a Sphere

2.1 Parametrisation

A vector is taken, whose length equals the radius of the sphere that we wish to parametrise, which, for now, we will call r, and, for the purpose of convenience, it is, arbitrarily, put along the e_1-axis.

By rotating it in the e_1, e_2-plane by an angle ϕ_1, it will point to anywhere on a circle of radius r centered in the origin, for $0 \leq \phi_1 \leq 2\pi$. By rotating it consequently in the e_1, e_3-plane by an angle ϕ_2, it will point along the entire sphere for $0 \leq \phi_2 \leq 2\pi$. There then is a parametrisation of the sphere and at the same time a method to uniquely specify any point on the sphere by two variables – the angles ϕ_1 and ϕ_2.

Using Geometric Algebra, it is possible to find an expression for these rotations. We can write it trigonometrically since $e^{\alpha e_i e_j} = \cos\alpha + (e_i e_j)\sin\alpha$ for $i, j > 0$ [1].

First, the following quantities are defined:

$$\begin{aligned} R_{12} &\equiv \cos\frac{\phi_1}{2} - e_1 e_2 \sin\frac{\phi_1}{2} \\ \tilde{R_{12}} &\equiv \cos\frac{\phi_1}{2} + e_1 e_2 \sin\frac{\phi_1}{2} \\ R_{13} &\equiv \cos\frac{\phi_2}{2} - e_1 e_3 \sin\frac{\phi_2}{2} \\ \tilde{R_{13}} &\equiv \cos\frac{\phi_2}{2} + e_1 e_3 \sin\frac{\phi_2}{2} \end{aligned} \tag{5}$$

This yields, for any point p on the sphere (substituting $C_{i/k} \equiv \cos\frac{\phi_i}{k}$, $S_{i/k} \equiv \sin\frac{\phi_i}{k}$)

$$\begin{aligned} p &= R_{13} R_{12} r e_1 \tilde{R_{12}} \tilde{R_{13}} \\ &= r(C_{2/2} - e_1 e_3 S_{2/2})(C_1 - S_1 e_1 e_2)(C_{2/2} + e_1 e_3 S_{2/2}) \\ &= r(C_1 C_{2/2}^2 e_1 + C_{2/2} C_1 S_{2/2} e_1 e_1 e_3 - S_1 C_{2/2}^2 e_2 - C_{2/2} S_1 S_{2/2} e_2 e_1 e_3 \\ &\quad -S_{2/2} C_1 C_{2/2} e_1 e_3 e_1 - S_{2/2}^2 C_1 e_1 e_3 e_1 e_1 e_3 + \\ &\quad S_{2/2} S_1 C_{2/2} e_1 e_3 e_2 + S_{2/2}^2 S_1 e_1 e_3 e_2 e_1 e_3) \\ &= r(C_1 C_{2/2}^2 e_1 - C_{2/2} C_1 S_{2/2} e_3 - S_1 C_{2/2}^2 e_2 - C_{2/2} S_1 S_{2/2} e_2 e_1 e_3 \\ &\quad -S_{2/2} C_1 C_{2/2} e_3 - S_{2/2}^2 C_1 e_1 + S_{2/2} S_1 C_{2/2} e_1 e_3 e_2 - S_{2/2}^2 S_1 e_2) \\ &= r(C_1 C_{2/2}^2 e_1 - C_{2/2} C_1 S_{2/2} e_3 - S_1 C_{2/2}^2 e_2 \\ &\quad -S_{2/2} C_1 C_{2/2} e_3 - S_{2/2}^2 C_1 e_1 - S_{2/2}^2 S_1 e_2) \\ &= r(C_1 C_2 e_1 - S_1 e_2 - C_1 S_2 e_3) \end{aligned}$$

This is in line what could be expected from the general rotations. For it can be written that $R_{12} e_1 \tilde{R_{12}} = \cos\phi_1 e_1 - \sin\phi_1 e_2$, where we define $x \equiv r\cos\phi_1$ and $y \equiv -r\sin\phi_1$. When a second rotation is applied, this time in the $e_1 e_3$-plane, then:

$$R_{13}(x e_1 + y e_2)\tilde{R_{13}} = x\cos\phi_2 e_1 + y e_2 - x\sin\phi_2 e_3$$

$$= r(\cos\phi_1\cos\phi_2 e_1 - \sin\phi_1 e_2 - \cos\phi_1\sin\phi_2 e_3) \quad (6)$$

2.2 Tangent Vectors

The tangent vectors are obtained by differentiating the parametrisation with respect to the parameters: $t_1 = \frac{\partial p}{\partial \phi_1}$ and $t_2 = \frac{\partial p}{\partial \phi_2}$.

Before computing them, it seems wise to establish the following.

The switching of $\tilde{R_{12}}$ and e_1e_2 is allowed since for $i \neq j$, $i, j > 0$ it holds that $e_ie_j\tilde{R_{ij}} = e_ie_j(\cos\frac{\phi_n}{2} + e_ie_j\sin\frac{\phi_n}{2}) = \cos\frac{\phi_n}{2}e_ie_j + e_ie_j\sin\frac{\phi_n}{2}e_ie_j = \tilde{R_{ij}}e_ie_j$. The same then also holds for R_{12}, since it differs only from $\tilde{R_{12}}$ in a minus sign in front of the sine.

The switching of $\tilde{R_{13}}$ and e_1e_2 generates a minus sign inside R_{ij} and therefore changes the pair of R_{ij} and $\tilde{R_{ij}}$ into one another, since $e_1e_2\tilde{R_{13}} = e_1e_2(\cos\frac{\phi_2}{2} + e_1e_3\sin\frac{\phi_2}{2} = \cos\frac{\phi_2}{2}e_1e_2 - e_1e_3\sin\frac{\phi_2}{2}e_1e_2) = R_{13}e_1e_2$.

Now computing the tangent vectors, the following expressions are obtained:

$$\begin{aligned}
t_1 &= \frac{\partial p}{\partial \phi_1} = R_{13}(R_{12}re_1\frac{\partial \tilde{R_{12}}}{\partial \phi_1} + \frac{\partial R_{12}}{\partial \phi_1}re_1\tilde{R_{12}})\tilde{R_{13}} \\
&= R_{13}(\frac{1}{2}R_{12}re_1e_1e_2\tilde{R_{12}} - \frac{1}{2}e_1e_2R_{12}re_1\tilde{R_{12}})\tilde{R_{13}} \\
&= \frac{1}{2}(R_{13}R_{12}re_1\tilde{R_{12}}e_1e_2\tilde{R_{13}} - R_{13}e_1e_2R_{12}re_1\tilde{R_{12}}\tilde{R_{13}}) \\
&= \frac{1}{2}(R_{13}R_{12}re_1\tilde{R_{12}}R_{13}e_1e_2 + R_{13}R_{12}re_1\tilde{R_{12}}R_{13}e_1e_2) \\
&= R_{13}R_{12}re_1\tilde{R_{12}}R_{13}e_1e_2 \qquad (7) \\
t_2 &= \frac{\partial p}{\partial \phi_2} = R_{13}(R_{12}re_1\tilde{R_{12}}\frac{\partial \tilde{R_{13}}}{\partial \phi_2}) + \frac{\partial R_{13}}{\partial \phi_2}(R_{12}re_1\tilde{R_{12}}\tilde{R_{13}}) \\
&= \frac{1}{2}(pe_1e_3 - e_1e_3p) = p \wedge (e_1e_3) \qquad (8)
\end{aligned}$$

The same tangent vectors can be obtained using the formulation in terms of sines and cosines:

$$t_1 = \frac{\partial p}{\partial \phi_1} = r(-\sin\phi_1\cos\phi_2 e_1 - \cos\phi_1 e_2 + \sin\phi_1\sin\phi_2 e_3) \qquad (9)$$

$$t_2 = \frac{\partial p}{\partial \phi_2} = r(-\cos\phi_1\sin\phi_2 e_1 - \cos\phi_1\cos\phi_2 e_3) \qquad (10)$$

2.3 Surface Area

To compute the surface integral of this shape, the length of the wedge vector of the two tangent vectors will be integrated along all possible values of the parameters to yield the surface.

$$\begin{aligned} \delta A &= t_1 \wedge t_2 \\ \|\delta A\| &= \sqrt{-(t_1 \wedge t_2)^2} \end{aligned} \tag{11}$$

Substituting the trigonometric formulation of the tangent vectors (in this case the simplest alternative) the following expression is obtained:

$$\begin{aligned} \frac{t_1 t_2}{r^2} &= (S_1 C_2 e_1 + C_1 e_2 - S_1 S_2 e_3)(C_1 S_2 e_1 + C_1 C_2 e_3) \\ &= -S_1 S_2 C_1 C_2 + S_1 C_1 C_2^2 e_1 e_3 + S_2 C_1^2 e_2 e_1 \\ &\quad + C_1^2 C_2 e_2 e_3 - S_1 S_2^2 C_1 e_3 e_1 + S_1 S_2 C_1 C_2 \\ &= S_1 C_1 e_1 e_3 + S_2 C_1^2 e_2 e_1 + C_1^2 C_2 e_2 e_3 \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{t_2 t_1}{r^2} &= (C_1 S_2 e_1 + C_1 C_2 e_3)(S_1 C_2 e_1 + C_1 e_2 - S_1 S_2 e_3) \\ &= -S_1 S_2 C_1 C_2 + S_2 C_1^2 e_1 e_2 - S_1 S_2^2 C_1 e_1 e_3 \\ &\quad + S_1 C_1 C_2^2 e_3 e_1 + C_1^2 C_2 e_3 e_2 + S_1 S_2 C_1 C_2 \\ &= S_2 C_1^2 e_1 e_2 - S_1 C_1 e_1 e_3 + C_1^2 C_2 e_3 e_2 \end{aligned} \tag{13}$$

It can then be established that

$$\frac{t_1 t_2 + t_2 t_1}{r^2} = 0 = \frac{2(t_1 \wedge t_2)}{r^2} \tag{14}$$

This means that $t_1 \cdot t_2 = 0$ and therefore $t_1 \wedge t_2 = t_1 t_2$ and, consequently, $t_1 t_2 = t_1 \wedge t_2 = -(t_2 \wedge t_1) = -t_2 t_1$.

It is then possible to rewrite the original equation for the surface element:

$$\|\delta A\| = \sqrt{-(t_1 t_2)^2} = \sqrt{-t_1 t_2 t_1 t_2} = \sqrt{t_1^2 t_2^2} \tag{15}$$

In computing the squares, it can be established that for $i, j, k > 0$ and $i \neq j \neq k \neq i$, it holds that $(c e_i e_j)^2 = -c^2 e_i e_i e_j e_j = -c^2$ and $(c e_i e_j)(d e_k e_i) = c d e_i e_j e_k e_i = -c d e_k e_i e_i e_j = -(d e_k e_i)(c e_i e_j)$. Due to this latter property the cross terms in the product $(t_1 t_2)^2$ vanish, and due to the former property the bivector products (but not their coefficients).

This results in:

$$\begin{aligned}
\|\delta A\| &= r^2\sqrt{S_1^2C_1^2 + S_2^2C_1^4 + C_1^4C_2^2} \\
&= r^2\sqrt{C_1^2(S_1^2 + C_1^2(S_2^2 + C_2^2))} \\
&= r^2\sqrt{(\cos\phi_1)^2} \qquad (16)
\end{aligned}$$

The surface area of the sphere is then given by the following integral:

$$S_s = \int_0^{2\pi}\int_0^{2\pi} \|\delta A\| d\phi_1 d\phi_2 \qquad (17)$$

However, this integral integrates over all possible angles, and thereby would go over the entire sphere twice. Since the first integral, over all angles ϕ_1, results in a circle. Integrating this result over all angles ϕ_2 from 0 to π, the surface of the sphere is complete. Integrating ϕ_2 from π to 2π is therefore superfluous and would result in twice the real surface area of the sphere. Rewriting the integral the following expression is obtained:

$$\begin{aligned}
S_s &= \int_0^{2\pi}\int_0^{2\pi} \|\delta A\| d\phi_1 d\phi_2 \\
&= r^2\int_0^{\pi}\int_0^{2\pi} \sqrt{(\cos\phi_1)^2} d\phi_1 d\phi_2 \\
&= r^2\int_0^{\pi} 4 d\phi_2 \\
&= 4\pi r^2 \qquad (18)
\end{aligned}$$

2.4 Volume

The volume is obtained by evaluating the integral of the surface of all possible spheres of small radii r' such that $0 \leq r' \leq r$. This results in the following integral:

$$\begin{aligned}
V_s &= \int_0^{r} 4\pi r'^2 dr' = \frac{4}{3}\pi r'^3|_0^r \\
&= \frac{4}{3}\pi r^3 \qquad (19)
\end{aligned}$$

3 Parametrising a Torus

3.1 Parametrisation

A vector is taken, of length r, and, for the purpose of convenience, it is, arbitrarily, put along the e_1-axis. By rotating it in the e_1, e_2-plane by an angle ϕ_1, it will point to anywhere on a circle of radius r centered in the origin, for $0 \leq \phi_1 \leq 2\pi$.

Next this new rotated vector is pushed outward in the $e_1 e_2$-plane, by adding a multiple of the e_1-base vector to it. This multiple is u and will be the radius of the torus. If $u = 0$ the situation is the same as before and it is a sphere that is parametrised.

By rotating it consequently in the e_1, e_3-plane by an angle ϕ_2, it will point, for the right choices of the coefficients anywhere on the torus, and $0 \leq \phi_2 \leq 2\pi$. There then is a parametrisation of the torus and at the same time a method to uniquely specify any point on it by two variables – the angles ϕ_1 and ϕ_2.

Using Geometric Algebra, it is possible to find an expression for these rotations. We can write it trigonometrically since $e^{\alpha e_i e_j} = \cos\alpha + (e_i e_j)\sin\alpha$ for $i, j > 0$.

The same definitions given in the set of equations 5 will be used.

This yields, for any point q on the torus (substituting $C_{i/k} \equiv \cos\frac{\phi_i}{k}$, $S_{i/k} \equiv \sin\frac{\phi_i}{k}$)

$$
\begin{aligned}
q &= R_{13}(R_{12} r e_1 \tilde{R_{12}} + u e_1)\tilde{R_{13}} \\
&= R_{13} R_{12} r e_1 \tilde{R_{12}} \tilde{R_{13}} + R_{13} u e_1 \tilde{R_{13}} \\
&= p + R_{13} u e_1 \tilde{R_{13}}
\end{aligned}
$$

This is in line what could be expected from the general rotations. For it can be written that $R_{12} e_1 \tilde{R_{12}} = \cos\phi_1 e_1 - \sin\phi_1 e_2$, where we define $x \equiv r\cos\phi_1 + u$ and $y \equiv -r\sin\phi_1$. When a second rotation is applied, this time in the $e_1 e_3$-plane, then:

$$
\begin{aligned}
q &= R_{13}(x e_1 + y e_2)\tilde{R_{13}} \\
&= x\cos\phi_2 e_1 + y e_2 - x\sin\phi_2 e_3 \\
&= r\cos\phi_1\cos\phi_2 e_1 + u\cos\phi_2 e_1 - r\sin\phi_1 e_2 \\
&\quad -r\cos\phi_1\sin\phi_2 e_3 - u\sin\phi_2 e_3 \\
&= r(\cos\phi_1\cos\phi_2 e_1 - \sin\phi_1 e_2 - \cos\phi_1\sin\phi_2 e_3) + u(\cos\phi_2 e_1 - \sin\phi_2 e_3)
\end{aligned}
$$

3.2 Tangent Vectors

To obtain the tangent vectors, the derivatives of the parametrisation r with respect to ϕ_1 and ϕ_2 will be computed using the rules for switching terms discovered in section 2.2:

$$
\begin{aligned}
t_1 &= \frac{\partial q}{\partial \phi_1} = \frac{\partial p}{\partial \phi_1} = R_{13}R_{12}re_1\tilde{R_{12}}R_{13}e_1e_2 && (20) \\
t_2 &= \frac{\partial q}{\partial \phi_2} = \frac{\partial p}{\partial \phi_2} + \frac{\partial R_{13}}{\partial \phi_2}ue_1\tilde{R_{13}} + R_{13}ue_1\frac{\partial \tilde{R_{13}}}{\partial \phi_2} \\
&= \frac{\partial p}{\partial \phi_2} + \frac{1}{2}(e_1e_3R_{13}ue_1\tilde{R_{13}} - R_{13}ue_1\tilde{R_{13}}e_1e_3) && (21) \\
&= (p \wedge e_1e_3) + (R_{13}ue_1\tilde{R_{13}} \wedge e_1e_3) = (p + R_{13}ue_1\tilde{R_{13}}) \wedge e_1e_3 && (22)
\end{aligned}
$$

The trigonometrical equivalent is as follows:

$$
\begin{aligned}
t_1 &= \frac{\partial q}{\partial \phi_1} = r(-\sin\phi_1\cos\phi_2e_1 - \cos\phi_1e_2 + \sin\phi_1\sin\phi_2e_3) = \frac{\partial p}{\partial \phi_1} && (23) \\
t_2 &= \frac{\partial q}{\partial \phi_2} = r(-\cos\phi_1\sin\phi_2e_1 - \cos\phi_1\cos\phi_2e_3) \\
&\quad +u(-\sin\phi_2e_1 - \cos\phi_2e_3) && (24)
\end{aligned}
$$

3.3 Surface Area

Denoting the tangent vectors of the sphere as t_{1s}, t_{2s} and substituting the tangent vector correcting the sphere to a torus $t_u \equiv u(-\sin\phi_2e_1 - \cos\phi_2e_3)$, it can be written that:

$$
\begin{aligned}
t_1 &= t_{1s} && (25) \\
t_2 &= t_{2s} + t_u && (26)
\end{aligned}
$$

And therefore

$$
\begin{aligned}
t_1t_2 + t_2t_1 &= t_{1s}(t_{2s}t_u) + (t_{2s}t_u)t_{1s} = t_{1s}t_{2s} + t_{1s}t_u + t_{2s}t_{1s} + t_ut_{1s} \\
&= t_{1s}t_u + t_ut_{1s} && (27)
\end{aligned}
$$

Elaborating $t_{1s}t_u$, the following expression is obtained:

$$
\frac{t_{1s}t_u}{ru} = (S_1C_2e_1 + C_1e_2 - S_1S_2e_3)(S_2e_1 + C_2e_3)
$$

$$
\begin{aligned}
&= S_1e_1e_3 + S_2C_1e_2e_1 + C_1C_2e_2e_3 && (28)\\
\frac{t_ut_{1s}}{ru} &= (S_2e_1 + C_2e_3)(S_1C_2e_1 + C_1e_2 - S_1S_2e_3)\\
&= S_2C_1e_1e_2 - S_1e_1e_3 + C_1C_2e_3e_2 && (29)
\end{aligned}
$$

Thus $t_{1s}t_u + t_ut_{1s} = 0 = t_1t_2 + t_2t_1$ and therefore $t_1 \wedge t_2 = t_1t_2$.

It is interesting to see that not only t_{1s} and t_{2s} are perpendicular (their dot product equals zero), but also t_{1s} and t_u. Furthermore, these vanishing sums of products is exactly the same as what happened in the case of the sphere. It is general and caused by the fact that, each of the products ce_ide_j and de_jcde_i necessarily become each other's negative, since c and d are scalars and e_i and e_j by definition anticommute.

It is therefore only necessary for the products cde_ie_i to cancel out. In other words, in the general multiplication $(ae_1 + be_2 + ce_3)(fe_1 + ge_2 + he_3)$ the cross terms (e_ie_j) will cancel the reverse product, therefore perpendicularity only requires an additional condition on the coefficients for the symmetric terms (e_ie_i), and it is in this case $af + bg + ch = 0$. This can be recognised as the inner product and requiring it to be zero for perpendicular vectors is equivalent to the non-Geometric Algebra vector calculus.

The surface element can now be written as:

$$\|\delta A\| = \sqrt{-(t_1t_2)^2} = \sqrt{-(t_{1s}t_{2s} + t_{1s}t_u)^2} \tag{30}$$

However, since t_u and t_{2s} are not generally perpendicular, it will not be wise to expand this square. Rather, an alternative approach is chosen.

Furthermore, rewriting the tangent vector corresponding to ϕ_2 yields $-t_2 = r(C_1S_2e_1 + C_1C_2e_3) + u(S_2e_1 + C_2e_3) = (rC_1 + u)S_2e_1 + (rC_1 + u)C_2e_3$. Substituting $m \equiv rC_1 + u$, it can be noted that $m^2 = r^2C_1^2 + 2ruC_1 + u^2$. The case of the sphere (for $u = 0$) it is evident that $m^2 = r^2(\cos\phi_1)^2$ and this was the result earlier on.

Then the squares of the tangent vectors can be computed:

$$
\begin{aligned}
-t_1^2 &= r^2(S_1C_2e_1 + C_1e_2 - S_1S_2e_3)^2 = r^2(S_1^2C_2^2 + C_1^2 + S_1^2S_2^2) = r^2\\
-t_1^2 &= ((rC_1 + u)S_2e_1 + (rC_1 + u)C_2e_3)^2 = m^2(S_2^2 + C_2^2) = m^2
\end{aligned}
$$

For the surface element $\|\delta A\|$, it holds that

$$\|\delta A\| = \sqrt{-t_1^2t_2^2} = \sqrt{r^2(rC_1 + u)^2} = r(rC_1 + u) \tag{31}$$

The integral over all possible values for the parameters yields:

$$
\begin{aligned}
S_t &= \int_0^{2\pi}\int_0^{2\pi} \|\delta A\| d\phi_1 d\phi_2 \\
&= \int_0^{2\pi}\int_0^{2\pi} r(r\cos\phi_1 + u) d\phi_1 d\phi_2 \\
&= \int_0^{2\pi} (r^2 \sin\phi_1|_0^{2\pi} + ru\phi_1|_0^{2\pi}) d\phi_2 \\
&= \int_0^{2\pi} (2\pi r u) d\phi_2 \\
&= 2\pi r u \phi_2|_0^{2\pi} \\
&= 4\pi^2 r u
\end{aligned} \tag{32}
$$

And this is a general expression for the surface area of any torus.

3.4 Volume

Interestingly, the same integral can be applied to finding the volume of the torus. It includes an additional integral that integrates the surface area over all tori of smaller radii r', $0 \le r' \le r$. Naturally, this will yield the volume.

The following expression is obtained:

$$
\begin{aligned}
V_t &= \int_0^{2\pi}\int_0^{2\pi}\int_0^{r} \|\delta A\| dr' d\phi_1 d\phi_2 \\
&= \int_0^{2\pi}\int_0^{2\pi}\int_0^{r} r'(r'\cos\phi_1 + u) dr' d\phi_1 d\phi_2 \\
&= \int_0^{2\pi}\int_0^{2\pi} (\frac{1}{3} r'^3 \cos\phi_1 + \frac{1}{2} r'^2 u)|_0^r d\phi_1 d\phi_2 \\
&= \int_0^{2\pi}\int_0^{2\pi} (\frac{1}{3} r^3 \cos\phi_1 + \frac{1}{2} r^2 u) d\phi_1 d\phi_2 \\
&= \int_0^{2\pi} \frac{1}{3} r^3 \sin\phi_1|_0^{2\pi} + \frac{1}{2} r^2 u \phi_1|_0^{2\pi} d\phi_2 \\
&= \int_0^{2\pi} r^2 u \pi d\phi_2 \\
&= r^2 u \pi \phi_2|_0^{2\pi} = 2\pi^2 r^2 u
\end{aligned} \tag{33}
$$

4 Conclusion

In conclusion, it has been shown in this paper that, using Geometric Algebra, expressions can be obtained for the parametrisation of the sphere and torus, and that these expressions can, in turn, be used to compute the surface area and volume of these elementary shapes.

The procedure is also general; for any regular (differentiable) surface it is conceivable that there can be constructed a parametrisation such that integrating over the wedge product of the tangent vectors obtained from differentiation of the parametrisation over the parameters results in the surface area of the shape.

Furthermore, Geometric Algebra has an important advantage over other computational methods and it is that there is a choice of representation: any vector could be notated trigonometrically or exponentially and the result would be the same.

References

[1] Floris van Vugt. On the application geometric algebra in the lorentz transformation. *University College Utrecht, The Netherlands*, 2003.