Paradoxes and Self–Reference

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1 Paradoxes

1.1 Truth and the problem of paradoxes

In the first approximation, a reasonable requirement on the definition of truth seems that a sentence such as "$\ulcorner p \urcorner$ is true"[1] has the same truth conditions as p itself, i.e.

$$\ulcorner p \urcorner \text{ is true} \leftrightarrow p, \tag{1}$$

which we call the T–equivalence for p.

Tarski's *Convention T* is the philosophical position that these, called T–equivalences, for all sentences p, not only derive from the conceptual content of "truth," but are also all there is to it[5](1).

The infamous Liar paradox consists in a sentence, called λ, which is "λ is false," i.e. $\lambda = \neg\mathrm{Tr}\ulcorner\lambda\urcorner$. It yields the following problem: if we substitute λ for p in the T–equivalence, we obtain (Tarski 1936):

$$\ulcorner \lambda \urcorner \text{ is true} \leftrightarrow \lambda \leftrightarrow \ulcorner \lambda \urcorner \text{ is false}$$

The problem is that if the above holds, then saying that λ is true, or that it is false, is equally problematic.

Formally, given a language $\mathcal{L}$, which is sufficiently strong, so that it can encode its own syntax in much the same way as Gödel encoded arithmetic

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[1] Here $\ulcorner p \urcorner$ represents the name of the sentence p. In the rest of this paper, for notational convenience p and $\ulcorner p \urcorner$ will be used interchangeably.

in arithmetic itself, then there can be no $\mathcal{L}$–formula $\mathrm{Tr}(x)$ which defines the set $\{\ulcorner\phi\urcorner | \phi \text{ is true in } \mathcal{L}\}$ (where $\ulcorner\phi\urcorner$ denotes the formula that encodes the sentence ϕ) i.e. a formula which is true of (the codes of) all true sentences and false of the false ones[13].

This is quite problematic, for truth is a fundamental concept in philosophical discourse, or the scientific enterprise for that matter. In particular, as McGee[10] remarks, it is difficult to imagine a concept more crucial to understanding the relationship between language and the outside world than truth. Therefore, it is imperative to find a concept of truth that is clear, coherent and that avoids paradoxicality such as that resulting from the liar sentence.[2]

1.2 Proposed solutions

1.2.1 Tarski

Tarski's solution is to start from the target language, L_0, and introduce another language, L_1 which contains a (coding of) all sentences and a *truth predicate* Tr_0 for L_0, i.e. a predicate that is true of all (codes of) sentences of L_0 and false of all other elements. Similarly, one can define L_2 which contains a truth predicate Tr_1 for L_1 and so on.

This makes formulation of λ impossible, for in this framework it would have to be of the form $\neg\mathrm{Tr}_n\ulcorner\lambda\urcorner$ for some $n \in \mathbb{N}$, which is, because of the occurrence of Tr_n, a sentence of L_{n+1}, therefore λ would have to be a sentence of L_n, which contradicts the occurrence of Tr_n.

However, this solution remains unsatisfying. From a linguistic point of view, it is counterintuitive that one would be talking in and switching between different languages in the proposed hierarchy, each of which does not contain a truth predicate for itself.

1.2.2 Kripke

On the basis of this observation, Kripke[8] endeavoured to find a way in which a language could contain its own truth predicate by adding a "neutral" truth value. That is, sentences are allowed to be, in addition to true or false, simply neither of the two. Kripke then shows that there is a possibility of constructing a truth predicate that satisfies all the T–equivalences. In

[2] There is an interesting parallel with Russel's Paradox that shook the foundations of set theory, which is a similarly crucial notion in mathematical reasoning. A solution was needed in the form of a new set theory that avoided the paradox and could yet be agreed upon by the mathematical community as a meaningful basis, a role Zermelo-Fraenkel eventually came to fulfill.

particular, λ is no longer problematic if we consider it neither true nor false. On the contrary, sentences that are true or false are said to be *grounded*.[3]

Clearly, Kripke is committed to explaining how a truth predicate can be defined so that contradictions like the liar paradox and all others are avoided.

1.2.3 Leitgeb

Another approach is offered more recently by Leitgeb[9] who retains the classical scheme (i.e., disallowing neutral truth values) but argues that the T–equivalences should not be required to hold for all sentences[7]. In particular, it should only hold for sentences that are said to be *grounded*.

Leitgeb's commitment is to provide a criterion that decides which T–equivalences hold, and in particular it is desired to be such that it rules out all the problematic sentences and leaves as many of the others in as possible. McGee[11] has shown that taking simply this restriction, the kinds of candidate sets of grounded sentences is virtually unrestricted. Therefore a more restrictive definitionis required, for which Leitgeb employs the notion of *dependence*.[4]

1.3 Problem statement

Informally, *groundedness* is based on a notion of reference[6], for grounded are those entences that either not contain the truth predicate at all, or where the sentences to which they apply the truth predicate can be traced back to non–semantic states of affairs: therefore, sentences that do not refer to themselves[14].

For instance, a sentence that says "$\ulcorner 2+2=4 \urcorner$ is true," formally $\mathrm{Tr}\ulcorner 2+2 = 4\urcorner$, refers to 2+2=4 which is a "non–semantic state of affairs," hence the sentence is grounded. Similarly, "$\ulcorner\ulcorner 2+2=4\urcorner$ is true$\urcorner$ is true,", refers to 2+2=4, although indirectly, but therefore it is also clearly grounded. On the other hand the liar sentence, λ, refers to itself, which implies an infinite chain of sentences that refer to the next that never reaches a ground, hence λ is not grounded.

Both Kripke's and Leitgeb's approach use this notion of *groundedness*. Therefore, one would expect that they agree about which sentences are *grounded*.

[3]More precisely, Kripke calls *grounded* those sentences that are true or false in the *least* fixed point (cf. section 2.1). This is essential, for sentences like the truth teller τ, which says "$\ulcorner\tau\urcorner$ is true," although not paradoxical, are intuitively not grounded, although they can take a truth value in a non–least fixed point[5].

[4]cf. section 2.2.

Although they agree about the above examples, Leitgeb[9][5] shows that this is not the case in general. The purpose of this present paper is to find which parameters cause this difference, so as to find by which adaptation of either approach an agreement can be reached.

2 Groundedness

Now the approaches of Leitgeb[9] and Kripke[8] will be looked at in formal detail.

2.1 Kripke

Given a classical[6] language $\mathcal{L}$ rich enough to allow its own syntax to be expressed in it. Let $i_{\mathcal{L}}$ interpret $\mathcal{L}$ into a domain D by the usual rules. $\mathrm{Tr}(x)$ will be the truth predicate, which will be interpreted by a partial function $i_{\mathrm{Tr}} : D \rightsquigarrow \{0,1\}$, to form the language $\mathcal{L}_{\mathrm{Tr}}$.

Supposing we have a set $E \subset D$ of (codes of) $\mathcal{L}_{\mathrm{Tr}}$–sentences that are considered true, and similarly $A \subset D$ for false sentences, then the $i_{\mathcal{L}}$ can be extended to cover all of $\mathcal{L}_{\mathrm{Tr}}$ in the following way:

$$i_{\mathcal{L}_{\mathrm{Tr}}(E,A)}(\mathrm{Tr})(d) = \begin{cases} 1 & \text{if } d \in E \\ 0 & \text{if } d \in A \\ \uparrow & \text{otherwise} \end{cases} \tag{2}$$

and the semantic values of other formulas involving Tr are defined by means of Kleene's strong three-valued logic.

Given $\mathcal{L}_{\mathrm{Tr}}(E,A)$ we can find

$$J_{(E,A)} \stackrel{\text{def}}{=} \{\phi \in \mathcal{L}_{\mathrm{Tr}} | \phi \text{ is true under } i_{\mathcal{L}_{\mathrm{Tr}}(E,A)}\} \tag{3}$$

$$J^-_{(E,A)} \stackrel{\text{def}}{=} \{\phi \in \mathcal{L}_{\mathrm{Tr}} | \phi \text{ is false under } i_{\mathcal{L}_{\mathrm{Tr}}(E,A)}\} \tag{4}$$

For notational convenience, given any set $E \subset \mathcal{L}_{\mathrm{Tr}}$ a "set of negatives" is defined: $\neg E \stackrel{\text{def}}{=} \{\phi | \neg\phi \in E\}$. Since $\mathcal{L}_{\mathrm{Tr}}(E,A)$ is a closed language, we find that $J^-_{(E,A)} = \neg J_{(E,A)}$.

If we generalise the above procedure we find a sequence $(E_\alpha)_{\alpha\in\mathrm{On}}$ as follows: $E_0 = \emptyset$, $E_{\alpha+1} = J_{(E_\alpha,\neg E_\alpha)}$ and $E_\beta = \bigcup_{\alpha<\beta} E_\alpha$.

The monotonicity of the sequence $(E_\alpha)_{\alpha\in\mathrm{On}}$ together with the given that the class of sentences is a set yields that there is a fixed point, hence a smallest one, which we will call E_∞.

[5] cf. section 2.3.

[6] i.e. two–valued

A sentence ϕ of $\mathcal{L}_{\mathrm{Tr}}$ is defined to be *grounded* if it has a truth value (i.e. true or false) in $\mathcal{L}_{\mathrm{Tr}}(E_\infty, \neg E_\infty)$. Hence ϕ is grounded iff $\phi \in E_\infty \cup \neg E_\infty$.

2.2 Leitgeb

Leitgeb employs a similar construction, with a classical language $\mathcal{L}$ that will be enriched with a two–valued truth predicate Tr to form $\mathcal{L}_{\mathrm{Tr}}$. If $\phi \in \mathcal{L}_{\mathrm{Tr}}$ then $\mathrm{Val}_\Psi(\phi)$ denotes the truth value in the standard model of arithmetic enriched with a truth predicate which has extension $\Psi \subset \mathcal{L}_{\mathrm{Tr}}$ (and due to totality the anti–extension is the complement of Φ). This is to say, Ψ is the set of sentences that we assume to be true.

We define that ϕ *depends* on $\Phi \subset \mathcal{L}_{\mathrm{Tr}}$ iff for all $\Psi_1, \Psi_2 \subset \mathcal{L}_{\mathrm{Tr}}$, we have that if $\mathrm{Val}_{\Psi_1}(\phi) \neq \mathrm{Val}_{\Psi_2}(\phi)$ then $\Psi_1 \cap \Phi \neq \Psi_2 \cap \Phi$. Then Leitgeb shows that $D_\phi \stackrel{\text{def}}{=} \{\Phi \subset \mathcal{L}_{\mathrm{Tr}} | \phi \text{ depends on } \Phi\}$ is a filter. If D_ϕ has a least element Φ, we say ϕ depends *essentially* on Φ.

Similarly $D^{-1}(\Phi) \stackrel{\text{def}}{=} \{\phi \in \mathcal{L}_{\mathrm{Tr}} | \phi \text{ depends on } \Phi\}$. Leitgeb shows D^{-1} to be monotonic, and for any Φ, $D^{-1}(\Phi)$ is an algebra.

We define an ordinal sequence $(\Phi_\alpha)_{\alpha\in\mathrm{On}}$ as follows: $\Phi_0 = \emptyset$, $\Phi_{\alpha+1} = D^{-1}(\Phi_\alpha)$ and $\Phi_\beta = \bigcup_{\alpha<\beta} \Phi_\alpha$. Due to Tarski's Fixed Point Theorem, there is a least fixed point of this sequence that we will call Φ_{lf}.

A sentence ϕ is *grounded* iff $\phi \in \Phi_{\mathrm{lf}}$.[7]

Clearly, the sequence $(\Phi_\alpha)_{\alpha\in\mathrm{On}}$ does not distinguish true and false sentences,therefore, Leitgeb[9](171) also introduces the derived sequence: $\Gamma_0 = \emptyset$, $\Gamma_{\alpha+1} = \{\phi \in \Phi_{\alpha+1} | \mathrm{Val}_{\Gamma_\alpha}(\phi) = 1\}$, and $\Gamma_\beta = \bigcup_{\alpha<\beta} \Gamma_\alpha$[8].

2.3 Comparison of Kripke and Leitgeb

However, Leitgeb[9](185) shows that of the sets of sentences grounded according to his notion and Kripke's, neither one includes the other. On the one hand, $\theta' \stackrel{\text{def}}{=} \mathrm{Tr}\ulcorner 2+2=4 \urcorner \vee \lambda$ is grounded according to Kripke but not according to Leitgeb, as can be shown by the same reasoning as in section 2.4.2. On the other hand, given the liar sentence λ, then one can define: $\theta \stackrel{\text{def}}{=} \mathrm{Tr}\ulcorner\lambda\urcorner \vee \neg\mathrm{Tr}\ulcorner\lambda\urcorner$, then it is grounded according to Leitgeb but not according to Kripke.

Clearly θ is grounded (and true) in Leitgeb's scheme, since for any predicate P in a two–valued scheme $P(x) \vee \neg P(x)$ holds, hence for any Φ we have

[7]Leitgeb[9](169,lemma 13) that ϕ is ungrounded if, but not only if, there exists a sequence $(\psi_n)_{n\in\mathbb{N}^*}$ with $\psi_n \in \mathcal{L}_{\mathrm{Tr}}$; $\psi_0 = \phi$ and for every $n \in \mathbb{N}$, Ψ_{n+1} generates $D(\psi_n)$ and $\psi_{n+1} \in \Psi_{n+1}$.

[8]But, as will be shown, this sequence is not the same as Cantini's, and they do not even lead to the same fixed point.

$\mathrm{Val}_\Phi(\theta) = 1$. Therefore θ depends on $\emptyset$ and also on Φ_{lf} in particular[9].

However, Kripke's system fails to make it grounded. Suppose *per assurdo* that $\theta \in \neg E_\infty$, this means that $\neg\theta \in E_\infty$. Then $\mathrm{Tr}\ulcorner\lambda\urcorner$ and $\neg\mathrm{Tr}\ulcorner\lambda\urcorner$, absurd. On the other hand, suppose $\theta \in E_\infty$. Since E_∞ is a fixed point of Φ, we have that θ is true in $\mathcal{L}_{\mathrm{Tr}}(E_\infty, \neg E_\infty)$, hence either $\mathrm{Tr}\ulcorner\lambda\urcorner$ or $\neg\mathrm{Tr}\ulcorner\lambda\urcorner$. This yields the usual paradox, since, crucially, all the T–biconditionals hold in Kripke's system, also for λ. Therefore, one finds $\mathrm{Tr}\ulcorner\lambda\urcorner \leftrightarrow \lambda \leftrightarrow \neg\mathrm{Tr}\ulcorner\lambda\urcorner$[10],[11]

2.4 Cantini

2.4.1 Cantini's supervaluation reformulation

These considerations lead Cantini[1] to propose a reformulation that yields the fixed point E'_∞ that includes θ. Instead of Kripke's approach, here one uses a classical, two–valued interpretation of Tr.

As in section 2.2, $\mathrm{Val}_\Psi(\phi)$ represents the truth value of the formula ϕ given that the Tr–predicate's extension is Ψ. A set $\Psi \subset \mathcal{L}_{\mathrm{Tr}}$ will be considered consistent if whenever $\psi \in \Psi$, then $\neg\psi \notin \Psi$. An operator is defined as, for all $\Phi \subset \mathcal{L}_{\mathrm{Tr}}$, $\mathrm{FV}(\Phi) \stackrel{\mathrm{def}}{=} \{\phi \in \mathcal{L}_{\mathrm{Tr}} | \forall\Psi \supset \Phi$, s.t. Ψ is consistent, $\mathrm{Val}_\Psi(\phi) = 1\}$, which is clearly monotonous. Crucially, if Φ is consistent, so is $\mathrm{FV}(\Phi)$.

A sequence $(E'_\alpha)_{\alpha\in\mathrm{On}}$ is defined as follows: $E'_0 = \emptyset$, $E'_{\alpha+1} = \mathrm{FV}(E'_\alpha)$ and $E'_\beta = \bigcup_{\alpha<\beta} E'_\alpha$. Its least fixed point is called E'_∞.

2.4.2 Cantini's reformulation remains more inclusive

Cantini's reformulation introduced in section 2.3 succesfully takes away all counterexamples like θ, for Leitgeb proves that $\Phi_{\mathrm{lf}} \subset E'_\infty \cup \neg E'_\infty$[9](185). However, the failure to obtain the converse inclusion, $\Phi_{\mathrm{lf}} \not\supset E'_\infty \cup \neg E'_\infty$, still holds because of the mentioned counterexample: $\theta' \stackrel{\mathrm{def}}{=} \mathrm{Tr}\ulcorner 2+2=4\urcorner \vee \lambda$.

First of all, Cantini grounds θ', because for any $\Phi \subset \mathcal{L}_{\mathrm{Tr}}$, $\mathrm{Val}_\Phi(\theta') = 1$, hence $\theta' \in \mathrm{FV}(\emptyset)$.[12]

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[9] Since Φ_{lf} does not have the closedness properties that languages have (in particular it is not said that if $\ulcorner p \vee q\urcorner \in \Phi_{\mathrm{lf}}$ then $\ulcorner p\urcorner \in \Phi_{\mathrm{lf}} \vee \ulcorner q\urcorner \in \Phi_{\mathrm{lf}}$), from $\theta \in \Phi_{\mathrm{lf}}$ we cannot deduce $\lambda \in \Phi_{\mathrm{lf}}$ or $\neg\lambda \in \Phi_{\mathrm{lf}}$.

[10] These biconditionals were not required to hold in Leitgeb's system for λ is ungrounded.

[11] A consequence of this reasoning is that Kripke's "grounded" is not closed under two–valued logical equivalence, for in the classical scheme, θ is logically equivalent to any other tautology, including those that do not involve the truth predicate and hence are grounded automatically. However, clearly Kripke's set of "grounded" sentences is closed under trivalent logical equivalence (where θ is no longer a tautology), i.e. $\phi \in \Phi_{\mathrm{lf}}$ and $\phi \leftrightarrow_3 \psi$ implies $\psi \in \Phi_{\mathrm{lf}}$. Leitgeb's set of "grounded" sentences, however, is closed also under bivalent logical equivalence.

[12] Also $\theta' \in E_\infty$. Clearly, since $\ulcorner 2+2=4\urcorner \in \mathcal{L}$, because we start from the model of

However, Leitgeb shows $\theta' \notin \Phi_{\mathrm{lf}}$. The point is that θ depends on $\{2+2 = 4, \lambda\}$ and essentially so: clearly $\mathrm{Val}_{\Phi \cap \{2+2=4,\lambda\}}(\theta) = \mathrm{Val}_{\Phi}(\theta)$ for any Φ and esentiality follows from that if θ depends on Φ, then $\ulcorner 2+2=4 \urcorner \in \Phi$ and similarly $\lambda \in \Phi$. But since $\lambda \notin \Phi_{\mathrm{lf}}$, θ does not depend on Φ_{lf} and is hence ungrounded.

2.5 Outline

The main problem adressed in this paper is what will be required on either side of the equation $\Phi_{\mathrm{lf}} \subsetneq E'_\infty \cup \neg E'_\infty$ to yield equality.

Section 3 will attempt to "raise" Φ_{lf} to $E'_\infty \cup \neg E'_\infty$ by presupposing a minimal extension of the predicate of truth in the notion of dependence. It will be shown in section 3.3.2 that there are still classes of sentences that belong to $E'_\infty \cup \neg E'_\infty$ but not to Φ_{lf}.

Section 4 shows that if one adds a consistency requirement to the definition of conditional dependence introduced in section 3 one obtains the same least fixed point $E'_\infty \cup \neg E'_\infty$ as Cantini.

These proceedings are summarised visually in appendix 6.

3 Conditional Dependence

3.1 Introduction

Following the discussion of section 2.4.2, it seems an undesirable situation that $\ulcorner 2+2=4 \urcorner, \mathrm{Tr}\ulcorner 2+2=4 \urcorner \in \Phi_{\mathrm{lf}}$ but $\mathrm{Tr}\ulcorner 2+2=4 \urcorner \vee \lambda \in \Phi_{\mathrm{lf}}$, for intuitively it seems the latter is "true" anyway in virtue of the truth of $\ulcorner 2+2=4 \urcorner$.

This reasoning leads to the definition of *conditional dependence*, suggested by Leitgeb[9](189), in which the attention is restricted to those Ψ_1, Ψ_2 that extend a set of sentences Σ that we presuppose as true. Given any $\Sigma \subset \mathcal{L}_{\mathrm{Tr}}$,

Definition 3.1. $\phi\, \mathrm{dep}_\Sigma(\Phi) \stackrel{\mathrm{def}}{=}$ for all $\Psi_1, \Psi_2 \subset \mathcal{L}_{\mathrm{Tr}}$ s.t. $\Sigma \subset \Psi_1, \Psi_2$ it holds that $\mathrm{Val}_{\Psi_1}(\phi) \neq \mathrm{Val}_{\Psi_2}(\phi) \rightarrow \Psi_1 \cap \Phi \neq \Psi_2 \cap \Phi$

3.2 General properties

3.2.1 Preliminaries

A number of general properties will now be established in order to show that the notion of conditional dependence functions along the same lines as

arithmetic, also $\ulcorner 2+2=4 \urcorner \in J_{(\emptyset,\emptyset)}$. Since we then close it as a language by Kleene's strong three–valued logic, where $\vee(t,n) = t$, the neutrality of λ doesn't keep θ' from being true. Therefore $\theta \in E_0$, hence $\theta \in E_\infty$.

dependence. These mirror Leitgeb[9](161)'s lemmas 2 and 3.

Lemma 3.1. *Under the crucial assumption that $\Phi \supset \Sigma$, are equivalent the following:*

1. ϕ *dep*$_\Sigma(\Phi)$
2. *For all* $\Sigma \subset \Psi \subset \mathcal{L}_{Tr}$, $Val_\Psi(\phi) = Val_{\Psi\cap\Phi}(\phi)$
3. $\forall \Psi_1, \Psi_2 \subset \mathcal{L}_{Tr}, \Sigma \subset \Psi_1, \Psi_2 \to (Val_{\Psi_1}(\phi) = Val_{\Psi_2}(\phi) \Leftrightarrow Val_{\Psi_1\cap\Phi}(\phi) = Val_{\Psi_2\cap\Phi}(\phi))$

Proof. $(1 \to 2)$. Taking any $\Psi \supset \Sigma$, then clearly $\Psi \cap \Phi \supset \Sigma$, so by $\phi\,\mathrm{dep}_\Sigma(\Phi)$ one finds $\mathrm{Val}_\Psi(\phi) = \mathrm{Val}_{\Psi\cap\Psi}(\phi)$.

$(2 \to 3)$. Given $\Psi_1, \Psi_2 \supset \Sigma$ then by 2, $\mathrm{Val}_{\Psi_1\cap\Phi}(\phi) = \mathrm{Val}_{\Psi_1}(\phi)$ and same for Ψ_2, so the equivalence follows.

$(3 \to 1)$. Suppose the contrary of 1, i.e. finding $\Psi_1, \Psi_2 \supset \Sigma$, $\mathrm{Val}_{\Psi_1}(\phi) \neq \mathrm{Val}_{\Psi_2}(\phi)$ but $\Psi_1 \cap \Phi = \Psi_2 \cap \Phi$. By the latter fact $\mathrm{Val}_{\Psi_1\cap\Phi}(\phi) = \mathrm{Val}_{\Psi_2\cap\Phi}(\phi)$, which contradicts the former because of 3. □

Lemma 3.2. *Filter properties of* *dep*$_\Sigma()$*:*

1. *If* ϕ *dep*$_\Sigma(\Phi), \Phi' \supset \Phi$ *then* ϕ *dep*$_\Sigma(\Phi')$
2. *If* ϕ *dep*$_\Sigma(\Phi)$, ϕ *dep*$_\Sigma(\Psi)$ *and* $\Phi \supset \Sigma$ *then* ϕ *dep*$_\Sigma(\Phi \cap \Psi)$
3. ϕ *dep*$_\Sigma(\mathcal{L}_{Tr})$

Proof. 2: Take any $\Psi_1, \Psi_2 \supset \Sigma$. Suppose $\Psi_1 \cap \Phi \cap \Psi = \Psi_2 \cap \Phi \cap \Psi$. By $\phi\,\mathrm{dep}_\Sigma(\Psi)$ and because $\Psi_1 \cap \Phi, \Psi_2 \cap \Phi \supset \Sigma$ we have $\mathrm{Val}_{\Psi_1\cap\Phi}(\phi) = \mathrm{Val}_{\Psi_2\cap\Phi}(\phi)$. On a different note with $\phi\,\mathrm{dep}_\Sigma(\Phi)$ we obtain $\mathrm{Val}_{\Psi_1}(\phi) = \mathrm{Val}_{\Psi_1\cap\Phi}(\phi)$ and $\mathrm{Val}_{\Psi_2}(\phi) = \mathrm{Val}_{\Psi_2\cap\Phi}(\phi)$, therefore $\mathrm{Val}_{\Psi_1}(\phi) = \mathrm{Val}_{\Psi_2}(\phi)$. Hence $\phi\,\mathrm{dep}_\Sigma(\Psi \cap \Psi)$. □

3.2.2 Formulation of the fixed point

Definition 3.2. $\mathrm{D}_\Sigma(\phi) \overset{\text{def}}{=} \{\Phi \subset \mathcal{L}_{\mathrm{Tr}} | \phi\,\mathrm{dep}_\Sigma(\Phi)\}$

$\mathrm{D}_{\supset\Sigma}(\phi) \overset{\text{def}}{=} \{\Phi \subset \mathcal{L}_{\mathrm{Tr}} | \Phi \supset \Sigma \wedge \phi\,\mathrm{dep}_\Sigma(\Phi)\}$

$\mathrm{D}^{-1}_\Sigma(\phi) \overset{\text{def}}{=} \{\ulcorner\phi\urcorner \in \mathcal{L}_{\mathrm{Tr}} | \phi\,\mathrm{dep}_\Sigma(\Phi)\}$

The reservations in lemma 3.2 make that it is no longer guaranteed that $\mathrm{D}_\Sigma(\phi)$ is a filter. However, its restriction to above Σ, $\mathrm{D}_{\supset\Sigma}(\phi)$, is.

Analogous to the sequence Φ_α defined by Leitgeb, one can introduce a parallel sequence $(\Phi^{\mathrm{AT}}_\alpha)_{\alpha\in\mathrm{On}}$ and $(\Gamma^{\mathrm{AT}}_\alpha)_{\alpha\in\mathrm{On}}$. The idea is again that one begins with the empty set, then takes all sentences that depend on it, and so on, but the difference with before is that at every step we presuppose (i.e. conditionalise) all grounded sentences that were true in the previous step.

Definition 3.3. $\Phi_0^{\mathrm{AT}} = \emptyset$, $\Gamma_0^{\mathrm{AT}} = \emptyset$, $\Phi_{\alpha+1}^{\mathrm{AT}} = \mathrm{D}^{-1}_{\Gamma_\alpha^{\mathrm{AT}}}(\Phi_\alpha^{\mathrm{AT}})$, $\Gamma_{\alpha+1}^{\mathrm{AT}} = \{\phi \in \Phi_{\alpha+1}^{\mathrm{AT}} | \mathrm{Val}_{\Gamma_\alpha^{\mathrm{AT}}}(\phi) = 1\}$, $\Phi_\beta^{\mathrm{AT}} = \bigcup_{\alpha<\beta} \Phi_\alpha^{\mathrm{AT}}$, $\Gamma_\beta^{\mathrm{AT}} = \bigcup_{\alpha<\beta} \Gamma_\alpha^{\mathrm{AT}}$,

Interestingly, one needs this "double recursion," where at every step in the expansion of the dependence set one presupposes all the truths of the previous step. If we would keep the conditional set Σ fixed for every instance of $\mathrm{D}^{-1}_\Sigma()$, for instance taking the set of arithmetical truths, AT, every time, then the problem rises that one captures the problematic $\mathrm{Tr}\ulcorner 2+2=4 \urcorner \vee \lambda$ but not $\mathrm{Tr}\ulcorner \mathrm{Tr}\ulcorner 2+2=4 \urcorner \urcorner \vee \lambda$.

Lemma 3.3. *For all $\Phi, \Phi', \Sigma, \Sigma' \subset \mathcal{L}_{Tr}$, for all $\alpha, \beta \in On$,*

1. *If $\Phi \subset \Phi'$ and $\Sigma \subset \Sigma'$ then $D^{-1}_\Sigma(\Phi) \subset D^{-1}_{\Sigma'}(\Phi')$*
2. *(a) $\Phi_\alpha^{AT} \subset \Phi_{\alpha+1}^{AT}$ and (b) $\Gamma_\alpha^{AT} \subset \Gamma_{\alpha+1}^{AT}$*

Proof. 2: proven in conjunction with (c) $\Gamma_{\alpha+1}^{\mathrm{AT}} \cap \Phi_\alpha^{\mathrm{AT}} = \Gamma_\alpha^{\mathrm{AT}}$ by induction on α. For $\alpha = 0$, all three statements are immediate. For $\alpha = \alpha' + 1$, (a) $\Phi_\alpha^{\mathrm{AT}} \subset \Phi_{\alpha+1}^{\mathrm{AT}}$ follows from 1.

To find (c): ($\subset$) If $\phi \in \Gamma_{\alpha'+1+1}^{\mathrm{AT}} \cap \Phi_{\alpha'+1}^{\mathrm{AT}}$ then also $\phi \, \mathrm{dep}_{\Gamma_{\alpha'}^{\mathrm{AT}}}(\Phi_{\alpha'}^{\mathrm{AT}})$. So $1 = \mathrm{Val}_{\Gamma_{\alpha'+1}^{\mathrm{AT}}}(\phi) = \mathrm{Val}_{\Gamma_{\alpha'+1}^{\mathrm{AT}} \cap \Phi_{\alpha'}^{\mathrm{AT}}}(\phi)$. Because of the induction hypothesis, $\mathrm{Val}_{\Gamma_{\alpha'+1}^{\mathrm{AT}} \cap \Phi_{\alpha'}^{\mathrm{AT}}}(\phi) = \mathrm{Val}_{\Gamma_{\alpha'}^{\mathrm{AT}}}(\phi)$, so $\phi \in \Gamma_{\alpha'+1}^{\mathrm{AT}}$.

($\supset$) If $\phi \in \Gamma_{\alpha'+1}^{\mathrm{AT}}$ hence $\mathrm{Val}_{\Gamma_{\alpha'}^{\mathrm{AT}}}(\phi) = 1$ and $\phi \in \Phi_{\alpha'+1}^{\mathrm{AT}}$. Therefore $\phi \, \mathrm{dep}_{\Gamma_{\alpha'}^{\mathrm{AT}}}(\Phi_{\alpha'}^{\mathrm{AT}})$ and then $\mathrm{Val}_{\Gamma_{\alpha'+1}^{\mathrm{AT}}}(\phi) = \mathrm{Val}_{\Gamma_{\alpha'+1}^{\mathrm{AT}} \cap \Phi_{\alpha'}^{\mathrm{AT}}}(\phi) = \mathrm{Val}_{\Gamma_{\alpha'}^{\mathrm{AT}}}(\phi)$ because of the induction hypothesis, and the latter equals 1.

Finally (b) follows from (c) directly. $\square$

Hence, the same argument as before shows that the sequence $(\Phi_\alpha^{\mathrm{AT}})_{\alpha \in \mathrm{On}}$ has a least fixed point, called $\Phi_{\mathrm{lf}}^{\mathrm{AT}}$.

3.2.3 Commentary: what about falsity?

It seems that the same argument that led from the observation of the failure to include phrases like $\mathrm{Tr}\ulcorner 2+2=4 \urcorner \vee \lambda$ to the notion of conditional dependence, could also lead from failure to include phrases like $\mathrm{Tr}\ulcorner 2+2 \neq 4 \urcorner \wedge \lambda$ to an adapted notion of dependence. This notion should not only specify which phrases we presuppose to be true, but also those that we presuppose to be false.

A candidate definition could be: $\phi_E^A \mathrm{dep}\ \Phi \stackrel{\mathrm{def}}{=} \forall \Psi_1, \Psi_2, E \subset \Psi_1, \Psi_2 \subset A^c : \mathrm{Val}_{\Psi_1}(\phi) \neq \mathrm{Val}_{\Psi_2}(\phi) \rightarrow \Psi_1 \cap \Phi \neq \Psi_2 \cap \Phi$.

However, if one adds a consistency requirement to the notion of dependence, as will be done in section 4, this example is evidently automatically taken care of as well.

3.3 Analysis

3.3.1 Conditionalisation of arithmetic

Clearly AT, the codes of all sentences that are true in the standard model of arithmetic, are contained in Γ_1^{AT}, which justifies the superscript AT, although, interestingly, the definition has never explicitly mentioned this set.

Also, problematic examples such as $\mathrm{Tr}\lceil 2+2=4\rceil \vee \lambda$ are now included in $\Phi_{\mathrm{lf}}^{\mathrm{AT}}$, for $\lceil 2+2=4\rceil \in \Gamma_{\mathrm{lf}}^{\mathrm{AT}}$ and $\mathrm{Val}_{\Psi}\left(\mathrm{Tr}\lceil 2+2=4\rceil \vee \lambda\right) = 1$ for any $\Psi \supset \{2+2=4\}$.

3.3.2 Conditional dependence does not equal Cantini

The question that will be of interest is whether all such problematic examples have been taken care of. In particular, is $\Phi_{\mathrm{lf}}^{\mathrm{AT}}$ equal to Cantini's least fixed point of section 2.4.1? This question will be answered negatively.

Given a formula $\psi \in \mathcal{L}_{\mathrm{Tr}}$, one defines $\sigma_\psi \stackrel{\mathrm{def}}{=} \mathrm{Tr}\lceil\psi\rceil \wedge \mathrm{Tr}\lceil\neg\psi\rceil$. The point is that σ_ψ expresses an inconsistency of the extension of Tr. In particular, $\sigma_\lambda = \mathrm{Tr}\lceil\lambda\rceil \wedge \mathrm{Tr}\lceil\neg\lambda\rceil$ expresses an inconsistency outside $\Phi_{\mathrm{lf}}^{\mathrm{AT}}$, for $\lambda \notin \Phi_{\mathrm{lf}}^{\mathrm{AT}}$.

3.3.3 $\sigma_\lambda \notin \Phi_{\mathbf{lf}}^{\mathbf{AT}}$

Lemma 3.4. *For any conditional subset $\Sigma \subset \mathcal{L}_{Tr}$, any $\Psi \subset \mathcal{L}_{Tr}$, and $\alpha \in On$,*

1. *If $\lambda \notin \Psi$ then $\lambda \notin D_\Sigma^{-1}(\Psi)$*
2. *$\lambda \notin \Phi_\alpha^{AT}$*
3. *$\sigma_\psi\ dep_\Sigma(\Psi) \leftrightarrow \{\psi, \neg\psi\} \subset \Psi$*
4. *$\sigma_\lambda \notin \Phi_\alpha^{AT}$*

Proof. 1: $1 = \mathrm{Val}_\Sigma(\lambda) \neq \mathrm{Val}_{\Sigma\cup\{\lambda\}}(\lambda) = 0$ but $\Sigma \cap \Psi = \Sigma$ (by assumption) and $(\Sigma \cup \{\lambda\}) \cap \Psi = \Sigma$ because $\lambda \notin \Psi$, so λ does not depend Σ–conditionally on Ψ. 2: if $\lambda \in \Phi_{\alpha'+1}^{\mathrm{AT}}$ then, since it depends on itself, $\lambda \in \Phi_{\alpha'}^{\mathrm{AT}}$ but $\lambda \notin \emptyset$. 3: $(\leftarrow)$ $\{\psi, \neg\psi\} \subset \Psi$ implies that σ_ψ depends on Ψ even without a conditional set, so in particular also with any Σ. $(\rightarrow)$ Suppose, $\psi \notin \Psi$ then we find $\mathrm{Val}_{\Psi\cup\{\neg\psi\}}(\sigma_\psi) = 0 \neq 1 = \mathrm{Val}_{\Psi\cup\{\neg\psi,\psi\}}(\sigma_\psi)$ although $(\Psi \cup \{\neg\psi\}) \cap \Psi = (\Psi \cup \{\psi, \neg\psi\}) \cap \Psi$. The case $\neg\psi \notin \Psi$ is symmetric. □

3.3.4 $\sigma_\lambda \in E'_\infty \cup \neg E'_\infty$

Lemma 3.5. *For any $\Psi, \Phi \subset \mathcal{L}_{Tr}$,*

1. *If Ψ is consistent, $Val_\Psi(\neg\sigma_\lambda) = 1$*

2. $\neg\sigma_\lambda \in FV(\Phi)$

3. $\sigma_\lambda \in \neg E'_\infty$

Proof. 1: Suppose $\mathrm{Val}_\Psi(\neg\sigma_\lambda) = 0$ then $\mathrm{Val}_\Psi(\sigma_\lambda) = 1$ hence $\lambda \in \Psi$ and $\neg\lambda \in \Psi$, absurd. □

Therefore, σ_λ is an example that is in $E'_\infty \cup \neg E'_\infty$ but not in $\Phi^{\mathrm{AT}}_{\mathrm{lf}}$, hence $E'_\infty \cup \neg E'_\infty \not\subset \Phi^{\mathrm{AT}}_{\mathrm{lf}}$.

3.4 Removing consistency requirement

The reasoning of section 3.3.4 shows that the consistency requirement in Cantini's formulation is the essential reason why σ_λ is always in the image of FV, so one could wonder whether if we take out that restriction, the newly obtained fixed point, $\pm E''_\infty$, equals Leitgeb's conditional fixed point $\Phi^{\mathrm{AT}}_{\mathrm{lf}}$. Instead of FV one then uses for $\Phi \subset \mathcal{L}_{\mathrm{Tr}}$, $\mathrm{FV}'(\Phi) \overset{\mathrm{def}}{=} \{\phi \in \mathcal{L}_{\mathrm{Tr}} | \forall\Psi \supset \Phi, \mathrm{Val}_\Psi(\phi) = 1\}$.

In this section it will be shown that this adaptation makes $\pm E''_\infty$ too exclusive, for then there are sentences like $\sigma_{2+2=4} = \mathrm{Tr}\ulcorner 2+2=4 \urcorner \wedge \mathrm{Tr}\ulcorner 2+2 \neq 4 \urcorner$, whichexpress an inconsistency *inside* $\Phi^{\mathrm{AT}}_{\mathrm{lf}}$, that are no longer included, although they are in $\Phi^{\mathrm{AT}}_{\mathrm{lf}}$.

3.4.1 $\sigma_{2+2=4} \in \Phi^{\mathbf{AT}}_{\mathbf{lf}}$

Lemma 3.6. *For all $\Sigma \subset \Psi \subset \mathcal{L}_{Tr}$,*

1. $\{2+2=4, 2+2\neq 4\} \subset D^{-1}_\Sigma(\Psi)$

2. *If* $\{2+2=4, 2+2\neq 4\} \subset \Phi$, *then* $\sigma_{2+2=4} \in D^{-1}_\Sigma(\Phi)$

3. $\sigma_{2+2=4} \in \Phi^{AT}_{lf}$

3.4.2 $\sigma_{2+2=4} \notin \pm E''_\infty$

Lemma 3.7. *For any consistent $\Phi \subset \mathcal{L}_{Tr}$,*

1. *If* $\{2+2=4, 2+2\neq 4\} \not\subset \Phi$, *then* $\sigma_{2+2=4} \notin FV'(\Phi)$.

2. $\{2+2=4, 2+2\neq 4\} \not\subset FV'(\Phi)$

3. *For all* $\alpha \in On$, $\sigma_{2+2=4} \notin E'_\alpha$

4. $\neg\sigma_{2+2=4} \notin FV'(\Phi)$

5. *For all* $\alpha \in On$, $\neg\sigma_{2+2=4} \notin E'_\alpha$

6. $\sigma_{2+2=4} \notin \pm E''_\infty$

Proof. 1: follows from $\mathrm{Val}_\Phi(\sigma_{2+2=4}) = 0$. 2: $\lceil 2+2 \neq 4 \rceil$ will never be in any $\mathrm{FV}'(\Phi)$ for it is false regardless of the extension of the Tr–predicate. 4: $\mathcal{L}_{\mathrm{Tr}} \supset \Phi$ and $\mathrm{Val}_{\mathcal{L}_{\mathrm{Tr}}}(\neg\sigma_{2+2=4}) \neq 1$. □

3.5 Role of the T–schema

3.5.1 $\mathbf{Tr}\lceil\psi\rceil \to \psi$

Following interest in the status of T–biconditionals, one can examine the status of the following family of formulas:

Definition 3.4. $\omega_\psi \stackrel{\mathrm{def}}{=} \mathrm{Tr}\lceil\psi\rceil \to \psi$

Behaviour of this phrase is determined by the exact contents of ψ. For instance, if $\psi \in \mathcal{L}$ and ψ is true, then clearly ω_ψ is a tautology in $\mathcal{L}_{\mathrm{Tr}}$, so it depends on $\emptyset$, and similarly if it is false then it depends on ψ.

Taking $\psi = \lambda$, we find that $\omega_\lambda = \mathrm{Tr}\lceil\lambda\rceil \to \lambda = \mathrm{Tr}\lceil\lambda\rceil \to \neg\mathrm{Tr}\lceil\lambda\rceil$ is equivalent to λ. Hence, following the reasoning of section 3.3.3, it depends essentially on $\{\lambda\}$, hence $\omega_\lambda \notin \Phi^{\mathrm{AT}}_{\mathrm{lf}}$.

Lemma 3.8. *For all consistent* $\Psi \subset \mathcal{L}_{Tr}$,

1. $Val_\Psi(\omega_\lambda) = 1 \leftrightarrow \lambda \notin \Psi$

2. $\omega_\lambda \in FV(\Psi) \leftrightarrow \neg\lambda \in \Psi \vee Tr\lceil\lambda\rceil \in \Psi$

3. $\neg\omega_\lambda \in FV(\Psi) \leftrightarrow \lambda \in \Psi$

4. $\phi \in E'_\infty \leftrightarrow Tr\lceil\phi\rceil \in E'_\infty$

5. $\omega_\lambda \notin E'_\infty \cup \neg E'_\infty$

Proof. 2: ($\to$) If the consequent is false, then $\Psi \cup \{\lambda\}$ is a consistent superset of Ψ but $\mathrm{Val}_{\Psi\cup\{\lambda\}}(\omega_\lambda) = 0$. 3: ($\to$) $\neg\omega_\lambda \in \mathrm{FV}(\Psi)$ requires $\mathrm{Val}_\Psi(\neg\omega_\lambda) = 1$ which can only be if $\lambda \in \Psi$. 4: ($\to$) taking any $\Psi \supset E'_\infty$ then by assumption $\phi \in \Psi$, hence $\mathrm{Val}_\Psi(\mathrm{Tr}\lceil\phi\rceil) = 1$. ($\leftarrow$) suppose $\phi \notin E'_\infty$, then $\mathrm{Val}_{E'_\infty}(\mathrm{Tr}\lceil\phi\rceil) = 0$ contrary to the assumption. 5: using that $\{\lambda, \neg\lambda\} \cap E'_\infty = \emptyset$. From $\lambda \notin E'_\infty$ follows, due to 4, $\mathrm{Tr}\lceil\lambda\rceil \notin E'_\infty$. □

To sum up, $\omega_\lambda \notin \Phi^{\mathrm{AT}}_{\mathrm{lf}}$ and $\omega_\lambda \notin E'_\infty \cup \neg E'_\infty$

3.5.2 Membership is not truth in E'_∞

Cantini[1] shows, ω_λ is true in E'_∞, although in section 3.5.1 it is shown ω_λ is not a member of E'_∞.

Therefore it becomes clear that "being true in E'_∞" is not the same thing as "belonging to E'_∞." The latter implies the former, because ϕ belonging to E'_∞ means ϕ being true under a truth predicate extending E'_∞, i.e. for all $\Psi \supset E'_\infty$, $\mathrm{Val}_\Psi(\phi) = 1$. However, the inverse is not the case. For instance, λ is true in E'_∞, since it is $\neg\mathrm{Tr}\ulcorner\lambda\urcorner$, but it is not a member of E'_∞. Similarly, $\mathrm{Tr}\ulcorner\lambda\urcorner \to \lambda$ is true but not $\mathrm{Tr}\ulcorner\lambda\urcorner \to \lambda \in E'_\infty$.

3.5.3 $\psi \to \mathbf{Tr}\ulcorner\psi\urcorner$

Definition 3.5. $\omega'_\psi \stackrel{\mathrm{def}}{=} \psi \to \mathrm{Tr}\ulcorner\psi\urcorner$

Again, the status of ω_ψ is determined by ψ. If $\psi \in \mathcal{L}$ and false, then ω_ψ depends on $\emptyset$, if true, then ω_ψ depends on $\{\psi\}$.

Reasoning analogous to lemma 3.8 leads to $\omega'_\lambda \notin \Phi^{\mathrm{AT}}_{\mathrm{lf}}$, $\omega'_\lambda \notin E'_\infty \cup \neg E'_\infty$.

However, the full T–schema for λ, i.e. $\omega_\lambda \wedge \omega'_\lambda$ is an outright contradiction, $\mathrm{Tr}\ulcorner\lambda\urcorner \leftrightarrow \neg\mathrm{Tr}\ulcorner\lambda\urcorner$, hence its negation is false under any extension of Tr[13] and therefore it is found in $E'_\infty \cup \neg E'_\infty$. Also, being an antilogy, it depends on $\emptyset$ and therefore $\omega_\lambda \wedge \omega'_\lambda \in \Phi_{\mathrm{lf}}, \Phi^{\mathrm{AT}}_{\mathrm{lf}}$.

4 Dependence with Consistency and Conditionality

4.1 Introduction

4.1.1 Preliminaries

The strategy of section 3.4 was to remove the consistency requirement in Cantini's formulation. However, it became too restrictive to yield equality with Leitgeb's $\Phi^{\mathrm{AT}}_{\mathrm{lf}}$.

The approach in this section is to add a requirement of consistency on the other side, that is, to Leitgeb's definition of conditional dependence, to arrive at what will be called *conditional c–dependence*.

Leitgeb[9](180) also considered adding this consistency requirement but decided not to so as to keep the theoretical assumptions of his notion of dependence minimal. It is nevertheless introduced here in order to find out if this is the missing ingredient for equality with Cantini's $E'_\infty \cup \neg E'_\infty$.

[13] In particular, not just in any consistent extension, but this is not used in the reasoning.

Definition 4.1. $\phi\,\mathrm{cdep}_\Sigma(\Phi) \stackrel{\text{def}}{=}$ for all *consistent* $\Psi_1, \Psi_2 \supset \Sigma : \mathrm{Val}_{\Psi_1}(\phi) \neq \mathrm{Val}_{\Psi_2}(\phi) \rightarrow \Psi_1 \cap \Phi \neq \Psi_2 \cap \Phi$.

Lemma 4.1. *Again, the following are equivalent, under the crucial assumptions that $\Phi \supset \Sigma$ and that Σ is consistent:*

1. *ϕ cdep$_\Sigma(\Phi)$*
2. *For all consistent $\Psi \supset \Sigma$, it holds that Val$_\Psi(\phi) =$ Val$_{\Psi\cap\Phi}(\phi)$*
3. *For all consistent $\Psi_1, \Psi_2 \subset \mathcal{L}_{Tr}$, such that $\Sigma \subset \Psi_1, \Psi_2$: Val$_{\Psi_1}(\phi) =$ Val$_{\Psi_2}(\phi) \Leftrightarrow$ Val$_{\Psi_1\cap\Phi}(\phi) =$ Val$_{\Psi_2\cap\Phi}(\phi)$*

Proof. ($1 \rightarrow 2$) Clearly Ψ and $\Psi \cap \Phi$ are consistent supersets of Σ, so the argument is the same as before. □

Lemma 4.2. *Filter properties of conditional consistent dependence, assuming that Σ is consistent:*

1. *If ϕ cdep$_\Sigma(\Phi), \Phi' \supset \Phi$ then ϕ cdep$_\Sigma(\Phi')$*
2. *If ϕ cdep$_\Sigma(\Phi)$, ϕ cdep$_\Sigma(\Psi)$ and $\Phi \supset \Sigma$ then ϕ cdep$_\Sigma(\Phi \cap \Psi)$*
3. *ϕ cdep$_\Sigma(\mathcal{L}_{Tr})$*

Definition 4.2. $\mathrm{D}_{\mathrm{c},\Sigma}(\phi) \stackrel{\text{def}}{=} \{\Phi \subset \mathcal{L}_{\mathrm{Tr}} | \phi\,\mathrm{cdep}_\Sigma(\Phi)\}$
$\mathrm{D}^{-1}_{\mathrm{c},\Sigma}(\phi) \stackrel{\text{def}}{=} \{\ulcorner\phi\urcorner \in \mathcal{L}_{\mathrm{Tr}} | \phi\,\mathrm{cdep}_\Sigma(\Phi)\}$

4.1.2 Fixed point construction

We define the parallel ordinal sequences exactly as in section 3.2.2, to grow to the least fixed point:

Definition 4.3. $\Phi^{\mathrm{c,AT}}_0 = \emptyset$, $\Gamma^{\mathrm{c,AT}}_0 = \emptyset$, $\Phi^{\mathrm{c,AT}}_{\alpha+1} = \mathrm{D}^{-1}_{\mathrm{c},\Gamma^{\mathrm{c,AT}}_\alpha}(\Phi^{\mathrm{c,AT}}_\alpha)$, $\Gamma^{\mathrm{c,AT}}_{\alpha+1} = \{\phi \in \Phi^{\mathrm{c,AT}}_{\alpha+1} | \mathrm{Val}_{\Gamma^{\mathrm{c,AT}}_\alpha}(\phi) = 1\}$, $\Phi^{\mathrm{c,AT}}_\beta = \bigcup_{\alpha<\beta} \Phi^{\mathrm{c,AT}}_\alpha$, $\Gamma^{\mathrm{c,AT}}_\beta = \bigcup_{\alpha<\beta} \Gamma^{\mathrm{c,AT}}_\alpha$

The proof of monotonicity is omitted, for it will be proven that the sequence equals Cantini's, which is already known to be monotonic. For the moment, one can assume a fixed point will be reached, called $\Phi^{\mathrm{c,AT}}_{\mathrm{lf}}$.

Interestingly, there is a redundancy in the double recursion, which makes that it could have been defined as a single one:[14]

Lemma 4.3. *For all $\alpha \in On$, $\Phi^{c,AT}_\alpha = \pm\Gamma^{c,AT}_\alpha$*

[14] This redundancy holds also for dependency without conditionality.

Proof. It will be used that $\neg\psi\,\mathrm{cdep}_\Sigma(\Phi) \leftrightarrow \psi\,\mathrm{cdep}_\Sigma(\Phi)$.

Taking $\alpha = \alpha' + 1$. ($\subset$) Suppose $\phi \in \Phi^{\mathrm{c,AT}}_{\alpha'+1}$ but $\phi \notin \Gamma^{\mathrm{c,AT}}_{\alpha'+1}$. This means that $\mathrm{Val}_{\Gamma^{\mathrm{c,AT}}_{\alpha'}}(\phi) = 0$, which implies that $\mathrm{Val}_{\Gamma^{\mathrm{c,AT}}_{\alpha'}}(\neg\phi) = 1$, hence $\neg\phi \in \Gamma^{\mathrm{c,AT}}_{\alpha'+1}$ so $\phi \in \neg\Gamma^{\mathrm{c,AT}}_{\alpha'+1}$. ($\supset$) Suppose $\phi \in \pm\Gamma^{\mathrm{c,AT}}_{\alpha'+1}$. If $\phi \in \Gamma^{\mathrm{c,AT}}_{\alpha'+1}$ then one is done, otherwise $\neg\phi \in \Gamma^{\mathrm{c,AT}}_{\alpha'+1}$. So $\neg\phi \in \Phi^{\mathrm{c,AT}}_{\alpha'+1}$, which means $\neg\phi\,\mathrm{cdep}_{\Gamma^{\mathrm{c,AT}}_{\alpha'}}(\Phi^{\mathrm{c,AT}}_{\alpha'})$, so one obtains $\phi \in \Phi^{\mathrm{c,AT}}_{\alpha'+1}$. $\square$

Corollary 4.4. *If $\Psi \supset \Gamma^{c,AT}_{lf}$ and Ψ is consistent, then $\Psi \cap \Phi^{c,AT}_{lf} = \Gamma^{c,AT}_{lf}$*

As a result, the following is an equivalent formulation using single recursion:

Definition 4.4. $\Gamma^{\mathrm{c,AT}}_0 = \emptyset$, $\Gamma^{\mathrm{c,AT}}_{\alpha+1} = \{\phi \in \mathrm{D}^{-1}_{\mathrm{c},\Gamma^{\mathrm{c,AT}}_\alpha}(\pm\Gamma^{\mathrm{c,AT}}_\alpha) | \mathrm{Val}_{\Gamma^{\mathrm{c,AT}}_\alpha}(\phi) = 1\} \overset{\mathrm{def}}{=} \Delta_c(\Gamma^{\mathrm{c,AT}}_\alpha)$, $\Gamma^{\mathrm{c,AT}}_\beta = \bigcup_{\alpha<\beta} \Gamma^{\mathrm{c,AT}}_\alpha$,

and here it seemed elucidating to introduce an operator $\Delta_c()$ to represent the recursion.

4.2 Comparison with previous paradigm

What results from the difference between the notion of conditional dependence of section 3 and the notion of conditional c–dependence that includes a consistency requirement?

Clearly $\phi\,\mathrm{dep}_\Sigma(\Phi) \to \phi\,\mathrm{cdep}_\Sigma(\Phi)$ but not the converse. In particular, it has been shown that $\sigma_\psi = \mathrm{Tr}\ulcorner\psi\urcorner \wedge \mathrm{Tr}\ulcorner\neg\psi\urcorner$ depends essentially on $\{\psi, \neg\psi\}$, but it c–depends on $\emptyset$, since it is always false if the extension of Tr is consistent.

4.3 Reconciliation of Leitgeb and Cantini fixed points

In this section it will be shown that $\Phi^{\mathrm{c,AT}}_{\mathrm{lf}} = E'_\infty \cup \neg E'_\infty$: that, informally speaking, indeed Kripke's notion of groundedness in Cantini's formulation is equal to groundedness on the basis of conditional c–dependence.

A key role in the proof is performed by lemma 4.5, which explains that Cantini's supervaluation operator FV on a set Φ can be identified with consistent Φ–conditional dependence on the set $\pm\Phi$. A similar result cannot be obtained with a notion of dependency that does not include consistency, for then the maximality of Φ in $\pm\Phi$ does not hold.

Lemma 4.5. *For all consistent $\Phi \subset \mathcal{L}_{Tr}$ and $\phi \in \mathcal{L}_{Tr}$,*
$\phi\ cdep_\Phi(\pm\Phi) \leftrightarrow \phi \in \pm FV(\Phi)$

Proof. ($\rightarrow$) take any consistent $\Psi \supset \Phi$. It will be shown that $\mathrm{Val}_\Psi(\phi) = \mathrm{Val}_\Phi(\phi)$ which is sufficient. Clearly $\Psi \cap \pm\Phi = \Phi$, because Ψ is consistent. Then $\Psi \cap \pm\Phi = \Phi \cap \pm\Phi = \Phi$, so the dependency yields $\mathrm{Val}_\Psi(\phi) = \mathrm{Val}_\Phi(\phi)$.

($\leftarrow$) if $\phi \in \pm\mathrm{FV}(\Phi)$, clearly given any consistent $\Psi \supset \Phi$ one finds $\mathrm{Val}_\Psi(\phi) = \mathrm{Val}_\Phi(\phi)$, which means $\phi \,\mathrm{cdep}_\Phi(\emptyset)$ so in particular one also finds $\phi \,\mathrm{cdep}_\Phi(\pm\Phi)$. $\square$

Corollary 4.6. *For any consistent* $\Phi \subset \mathcal{L}_{Tr}$, $\Delta_c(\Phi) = FV(\Phi)$

Proof. By definition $\Delta_c(\Phi) = \{\phi \in \mathrm{D}^{-1}_{\mathrm{c},\Phi}(\pm\Phi) | \mathrm{Val}_\Phi(\phi) = 1\}$. Using lemma 4.5, one can rewrite $\Delta_c(\Phi) = \{\phi \in \pm\mathrm{FV}(\Phi) | \mathrm{Val}_\Phi(\phi) = 1\}$ which again equals $\{\phi \in \mathrm{FV}(\Phi) | \mathrm{Val}_\Phi(\phi) = 1\} \cup \{\phi \in \neg\mathrm{FV}(\Phi) | \mathrm{Val}_\Phi(\phi) = 1\}$. The first term $\{\phi \in \mathrm{FV}(\Phi) | \mathrm{Val}_\Phi(\phi) = 1\} = FV(\Phi)$, for if Φ is consistent, then $\phi \in \mathrm{FV}(\Phi)$ implies $\mathrm{Val}_\Phi(\phi) = 1$. Similarly, the second term $\{\phi \in \neg\mathrm{FV}(\Phi) | \mathrm{Val}_\Phi(\phi) = 1\} = \emptyset$, for $\neg\phi \in \mathrm{FV}(\Phi)$ implies $\mathrm{Val}_\Phi(\phi) = 0$. $\square$

Theorem 4.7. *For all* $\alpha \in On$, $\Phi^{c,AT}_\alpha = \pm E'_\alpha$ *and* $\Gamma^{c,AT}_\alpha = E'_\alpha$.

Proof. Immediate from defintion 4.4, lemma 4.3 and corollary 4.6. $\square$

Corollary 4.8. $\Phi^{c,AT}_{lf} = E'_\infty \cup \neg E'_\infty$

It is interesting to note that a stronger result has been proven than initially set out for: namely that not only the fixed points arrived at by Cantini and this dependency notion adapted from Leitgeb are the same, but also that every step of their construction is equal. This could lead to identify the which elements of these constructions are each other's counterparts.

4.4 Only consistency

One could ask if only adding the consistency requirement to Leitgeb's notion of dependence would have been enough to reach equivalence with Cantini's. However, sentences like $\theta' = \mathrm{Tr}\ulcorner 2+2=4 \urcorner \vee \lambda$ will still not be grounded, although they are so according to Cantini's notion[15].

5 Conclusion

5.1 Review

In this paper the notions of *groundedness* as introduced by Kripke and Leitgeb have been compared.

[15] cf. section 2.4.2.

It is shown that only when adding both a *conditionality* requirement and a *consistency* requirement at the same time to Leitgeb's notion of dependence that the resulting set of grounded sentences becomes identical to those following from Cantini's reformulation of Kripke's notion of groundedness.

An interesting question is what causes this equality. In particular, where exactly in Cantini's formulation does one find the counterparts of conditionality and consistency as used in the definition of dependency? For consistency the answer is not as clear as it seems, for although it appears overtly as a similar construction, it does not play the same role in Cantini's formulation, for section 3.4 shows removing it there yields a more restrictive notion of groundedness than Leitgeb's.

Widening our horizon, does this convergence reasonable suggest that we have found "the one right" set of grounded sentences? This remains to be seen, for it is not unimaginable that a similar reasoning could adapt Cantini's formulation to equal Leitgeb's original notion of dependency. On the other hand, the fact mentioned above that similar concepts do not play the same role in both formalisms indicates that we are faced with a more intricate interplay of parameters that perhaps only in a few, or even one rare constellation provide coherent notion of groundedness.

5.2 Future research

5.2.1 Necessary groundedness

As mentioned above, it would be interesting to see if the adaptations of Leitgeb's notion of dependency can be inversely applied to Cantini's so as to "lower" it to eventually equal Leitgeb's original formulation.

As part of this project such has also been attempted, taking the intersection of the fixed points resulting from Cantini's supervaluation based on any, not just the standard, interpretation of the arithmetical language. Formulated as such, it failed for exactly the same reason as conditional dependence failed,[16] but in the future a better formulation might be found.

5.2.2 Aboutness

Furthermore, if indeed we are faced with a correct notion of groundedness, it is expected to be a special case of a general theory of "aboutness"[12][3][4][2], in which groundedness would be "about non–semantic states of affairs." It would be interesting to see how precisely one obtains Leitgeb's and Cantini's formulation from such a theory.

[16] That is, the existence of σ_λ as in section 3.3.2.

References

[1] Andrea Cantini. A theory of formal truth arithmetically equivalent to id_1. *Journal of Symbolic Logic*, 55:244–259, 1990.

[2] Robert Demolombe and Luis Fariñas del Cerro. Toward a logical characterisation of sentences of the kind "sentence p is about object c". In S. Holldobler, editor, *Papers in honor of Wolfgang Bibel*, Intellectics and Computational Logic. Kluwer, 1999.

[3] Nelson Goodman. About. *Mind*, 70:1–24, 1961.

[4] Nelson Goodman. 'about' mistaken. *Mind*, 74:248, 1965.

[5] Anil Gupta and Nuel Belnap. *The Revision Theory of Truth*. MIT Press, Cambridge, Massachusetts, and London, England, 1993.

[6] Hans G. Herzberger. Paradoxes of grounding in semantics 145. *Journal of Philosophy*, 67:145–167, 1970.

[7] Paul Horwich. *Truth*. Oxford University Press, 1998.

[8] Saul Kripke. Outline of a theory of truth. *The Journal of Philosophy*, 72(19):690–716, Nov. 1975.

[9] Hannes Leitgeb. What truth depends on. *Journal of Philosophical Logic*, 34:155–192, 2005.

[10] Vann McGee. *Truth, vagueness, and paradox: an essay on the logic of truth*. Hackett Publishing Company, Indianapolis, 1991.

[11] Vann McGee. Maximal consistent sets of instances of tarski's schema (t). *Journal of Philosophical Logic*, 21:235–241, 1992.

[12] Hilary Putnam. Formalization of the concept 'about'. *Philosophy of Science*, 25:125–130, 1958.

[13] Alfred Tarski. The concept of truth in formalized languages. In J. Corcoran, editor, *Logic, Semantics and Metamathematics*. Hackett Publishing Company, Indianapolis, 1983. The English translation of Tarski's 1936 Der Wahrheitsbegriff in den formalisierten Sprachen.

[14] Stephen Yablo. Paradox without self-reference. *Analysis*, 53(4):251–252, Oct. 1993.

6 Appendix: Graphical representation