On the Application Geometric Algebra in the Lorentz Transformation

Floris T. van Vugt
University College Utrecht
Utrecht University, The Netherlands

September 26, 2003

Abstract

The aim of this paper is to provide understanding of the relation between the application of Geometric Algebra in the Lorentz Transformation and the conventional coordinate-based approach. The relation between the variables used in both approaches will be assessed and the relative efficiency of the Geometric Algebra will be established. The general Lorentz Transformation will be subdivided into a rotation and a boost transformation. This opens the way to write the exponent notation of the Geometric Algebra product as a Taylor series to discover the sine-cosine or hyperbolic sine-cosine alternative notation, which can successively be used to find the old notation. The power of the Geometric Algebra, it will be assessed, lies in the fact that it can solve the Lorentz Transformation without choosing any coordinate system.

1 Introduction

In computing Lorentz Transformation of complex physical system often many laborous computations can be avoided by choosing the right coordinate system. It would therefore appear *a priori* useful to be able to model a system to the greatest possible extent – so as to discover what conditions the optimal choice of coordinate axes satisfy – and only then having to choose a coordinate system.

Geometric Algebra is a system which enables us to do just that; to gain an understanding of the factors that contribute to unnecessary complications and then allowing the physicist to choose the coordinate system, in terms of the base vectors $e_k, k = 0, 1, 2, \cdots, n$ for any number n of dimensions.

However, the new formulation of the Lorentz Transformation leads to an alternative computation which at first glance seems dissimilar from the initial formulation. In this paper, the topic of discussion is how the formulation of the Lorentz Transformation as found in most textbooks relates to the equivalent in terms of Geometric Algebra.

1.1 Outline

The remainder of this article is organized as follows. In section 1 the hope is to come to a clear understanding of the problem area. Some of the properties of the geometric algebra and its geometric product will be discussed. Furthermore, both the intial formulation of the Lorentz Transformation and the Geometric Algebra equivalent will be discussed and some early comparisons will be drawn. Furthermore, the problem will be subdivided into two subproblems: Lorentz rotations and Lorentz boosts. In section 2 the problem will be solved using the Taylor expansion of the terms found in the Geometric Algebra equations, and then, writing these terms as sines and cosines or hyperbolic sines and cosines, it will be shown that they are equivalent to the initial formulation of the Lorentz Transformation.

1.2 Notation

In this paper, bivectors will be notated using the wedge, such that, for vectors a, b, c:

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c \qquad \text{and} \qquad a \wedge b = -b \wedge a \tag{1}$$

Also, the geometric product will be assumed to be defined as the sum of the inner (dot-) and outer (wedge-)product:

$$ab = a \cdot b + a \wedge b \tag{2}$$

Then

$$\begin{aligned} ba &= b \cdot a + a \wedge b \\ &= a \cdot b - a \wedge b \end{aligned}$$

And

$$a \cdot b = \frac{1}{2}(ab + ba) \qquad a \wedge b = \frac{1}{2}(ab - ba) \tag{3}$$

Orthonormal vectors will be used and denoted as e_n, with e_0 corresponding to the time-dimension. Therefore the following, by definition, holds (note that, due to their obvious vector nature, any arrows or bold face in these variables will, for convenience of notation and reading, be omitted):

$$\begin{aligned} e_i \cdot e_j &= 0 \qquad \text{for } i \neq j \\ e_i \cdot e_i &\equiv 1 \qquad \text{for } i = 0 \\ e_i \cdot e_i &\equiv -1 \qquad \text{for } i > 0 \end{aligned} \tag{4}$$

These properties are defined such that they amount to exactly what the *metric tensor* does in relativity without Geometric Algebra.

Therefore, it holds that:

$$e_i e_j = e_i \wedge e_j = -e_j e_i \tag{5}$$

This result will be referred to as *anticommutativity*, and will be of importance to the discussion later on.

1.3 Formulation of the problem

In this section it is tried to come to a formulation of the problem.

The Lorentz Transformation is used to convert the coordinates of the frame of reference of one observer Σ with the frame of reference of another, Σ'.

We will restrict our attention to the three space dimensions and one time dimension, which enables us to represent their coordinate systems in the vectors [Sch72]:

$$\Sigma \equiv \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} \qquad \text{and} \qquad \Sigma' \equiv \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} \tag{6}$$

The Lorentz transformation equations can be combined in the form of a $n \times n$ (n being the dimensions of the system, time included) matrix, which will here be arbitrarily called L, that acts on the vectors Σ to produce Σ':

$$\Sigma' = L\Sigma \tag{7}$$

This has provides us with an algorithm of a finite number of steps to find the coordinates of an event according to one frame of reference, if the coordinates according to another, equivalent, frame of reference are known. This is so by the nature of the Lorentz Transformation.

Geometric Algebra provides us with an algorithm to the same goal, but in a different form. It can be formulated as follows (α being a number usually referred to as "rapidity"):

$$x' = e^{-\frac{\alpha}{2}e_i e_j} x e^{\frac{\alpha}{2}e_i e_j} \tag{8}$$

In this case both x and x' being vectors that can be defined in the following way:

$$x \equiv cte_0 + xe_1 + ye_2 + ze_3 \quad x' \equiv ct'e_0 + xe_1 + ye_2 + ze_3 \tag{9}$$

The question that will be set out to answer in this paper is how these two different equations for the same thing relate to each other. It is apparent that both must lead to the same result, but in what exactly lies their analogy?

1.4 Division in simple problems

It will be held here that in problem solving, it is wise to break the problem up into simpler problems by excluding trivial complications that do not entail any fundamentalities. In the case of this particular problem such a simplification is thought to be made by distinguishing between different applications of the Taylor Transformation.

First we demarcate the domain of the problem by making the following assumptions:

- The coordinate axes of one frame of reference are parallel to the corresponding axes of the frame of reference to which, or from which, we switch.
- The two frames of reference are equivalent frames of reference (i.e. their relative velocity is constant and possibly zero). (This is a general assumption underlying the Lorentz Transformation).

Now we will distinguish between two applications of the Lorentz Transformation [CW98]:

- **Rotation along axis**: the frame of reference Σ' is rotated along two of the three space axes with respect to the frame of reference Σ.

- **Boost in a direction**: the frame of reference Σ' is moving with an arbitrary velocity in an arbitrary direction with respect to the frame of reference Σ.

1.5 Rotation

The Lorentz transformation in their initial formulation for a rotation along the x, y-axis over an angle θ can be established as follows [CW98]:

$$L = \begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \\ z' = z \\ ct' = ct \end{cases}$$

In the Geometric Algebra equivalent, this can be written as follows:

$$x' = e^{-\frac{\alpha}{2}e_1e_2} x e^{\frac{\alpha}{2}e_1e_2} \tag{10}$$

In the general discussion later on, this will be generalised to a rotation along any two base space vectors e_i, e_j.

1.6 Boost

Similarly, a boost along the x-axis (where the frame of reference Σ' has a speed v relative to the frame of reference Σ can be described by the following set of equations [CW98]:

$$L = \begin{cases} x' = -ct\sinh\theta + x\cosh\theta \\ y' = y \\ z' = z \\ ct' = ct\cosh\theta - x\sinh\theta \end{cases}$$

where $\frac{v}{c} = \tanh\theta$, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \cosh\theta$

Taking the equivalent from Geometric Algebra:

$$x' = e^{-\frac{\alpha}{2}e_0e_1} e_0 e^{\frac{\alpha}{2}e_0e_1} \tag{11}$$

2 Solution to the Problem

2.1 Taylor expansion

Before we go deeper into this matter, it could be fruitful to first discuss some of the properties of the term $e^{\alpha e_i e_j}$ (assuming $i \neq j$ and incorporating the denominator 2 in α) that is introduced in the Geometric Algebra equivalent of the Lorentz Transformation. First of all, however, it will be useful to know the properties of the form $(e_i e_j)^2$ for $i \neq j$:

$$\text{for} \quad i, j > 0 \qquad (e_i e_j)^2 = e_i e_j e_i e_j = -e_i e_i e_j e_j = -1 \tag{12}$$
$$\text{for} \quad i > 0, j = 0 \qquad (e_i e_0)^2 = -e_i e_0 e_0 e_i = 1 \tag{13}$$

If we now compute the Taylor series expansion of the function $e^{\alpha e_i e_j}$ for $i, j > 0$ and $i \neq j$ (therewith at this stage excluding time from the derivation) we see that:

$$\begin{aligned}
e^{\alpha e_i e_j} &= \sum_{n=0}^{\infty} \frac{\alpha^n (e_i e_j)^n}{n!} \\
e^{\alpha e_i e_j} &= 1 + \alpha e_i e_j + \frac{1}{2}\alpha^2 (e_i e_j)^2 + \frac{1}{6}\alpha^3 (e_i e_j)^3 + \ldots + \frac{1}{n!}\alpha^n (e_i e_j)^n \\
&= \sum_{n=0}^{\infty} \frac{\alpha^{2n} (e_i e_j)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\alpha^{2n+1} (e_i e_j)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{\alpha^{2n} (-1)^n}{(2n)!} + \sum_{n=0}^{\infty} \frac{\alpha^{2n+1} (-1)^n}{(2n+1)!} (e_i e_j) \\
&= \cos\alpha + (e_i e_j) \sin\alpha
\end{aligned} \tag{14}$$

From this, we also see the similarity in effect between the $e_i e_j$ term and the imaginary number i. Furthermore, we see that we are left with a peculiar quantity: the sum of a number ($\cos\alpha$) and a bivector ($(e_i e_j)\sin\alpha$). In this paper, this result will therefore afterwards be referred to as the *scalar-bivectorcomplex*, explicitly using the word *complex* to refer to the analogy with complex numbers.

In the case of $i > 0, j = 0$ we find that due to the result of equation 13 there is no alternation of signs and therefore:

$$e^{\alpha e_i e_0} = \sum_{n=0}^{\infty} \frac{\alpha^{2n} (e_i e_0)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\alpha^{2n+1} (e_i e_0)^{2n+1}}{(2n+1)!}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!}(e_i e_0) \\
&= \cosh \alpha + (e_i e_0) \sinh \alpha \qquad (15)
\end{aligned}
$$

This quantity is again a scalar-bivectorcomplex.

2.2 Scalar-bivectorcomplex numbers

The properties of the newly introduced scalar-bivectorcomplex numbers will be of importance later on and therefore be discussed prior to further inquiry in the subject matter.

Since the hyperbolic cosine is even ($\cosh -\alpha = \cosh \alpha$) and the hyperbolic sine is odd ($\sinh -\alpha = \sinh \alpha$), we obtain, when repeating the derivation in section 2.1, replacing α by $-\alpha$:

$$
\begin{aligned}
e^{-\alpha e_i e_j} &= \cos(\alpha) - (e_i e_j) \sin(\alpha) \qquad (16) \\
e^{-\alpha e_i e_0} &= \cosh(\alpha) - (e_i e_0) \sinh(\alpha) \qquad (17)
\end{aligned}
$$

These numbers, and their corresponding counterparts derived in section 2.1 will be referred to as *scalar-bivectorcomplex conjugates*. For notational convenience the product $e_i e_j$ for $i \neq j$ and $i > 0$ will be abbreviated as ε_j (such that it always holds that $\varepsilon_j^2 = -1$ and $\varepsilon_0^2 = 1$).

Multiplication of conjugates in the case $i, j > 0$ yields:

$$
\begin{aligned}
e^{-\alpha e_i e_j} e^{\alpha e_i e_j} &= (\cos \alpha - \varepsilon_j \sin \alpha)(\cos \alpha + \varepsilon_j \sin \alpha) \\
&= \cos^2 \alpha - \varepsilon_j^2 \sin^2 \alpha - \varepsilon_j \sin \alpha \cos \alpha + \varepsilon_j \sin \alpha \cos \alpha) \\
&= \cos^2 \alpha + \sin^2 \alpha = 1 \qquad (18)
\end{aligned}
$$

Similarly, in the case $i > 0, j = 0$, we find:

$$
\begin{aligned}
e^{-\alpha e_i e_0} e^{\alpha e_i e_0} &= (\cosh \alpha - \varepsilon_0 \sinh \alpha)(\cosh \alpha + \varepsilon_0 \sinh \alpha) \\
&= \cosh^2 \alpha - \varepsilon_0^2 \sinh^2 \alpha - \varepsilon_0 \sinh \alpha \cosh \alpha + \varepsilon_0 \sinh \alpha \cosh \alpha) \\
&= \cosh^2 \alpha - \sinh^2 \alpha = 1 \qquad (19)
\end{aligned}
$$

These properties will be of importance later on.

We have seen that $\varepsilon_0^2 = -1$ and this shows how similar ε_0 is to the imaginary unit i. From this perspective, the quantity α can be understood as corresponding to an angle, when written in the exponent in $e^{\frac{\alpha}{2}\varepsilon_0}$.

2.3 Lorentz Rotations

To come to a full understanding in the exact relationship it is at this point wise to find out how the different factors contribute to the problem. This is achieved by substituting equations 14, 15, 16 and 17 into equation 8 (taking the case $i, j > 0$ – corresponding to a rotation along the e_i, e_j-axis –, incorporating the $\frac{1}{2}$ in α and abbreviating $\cos\alpha$ and $\sin\alpha$ into C and S, respectively):

$$\begin{aligned}
x' &= e^{-\alpha e_i e_j} x e^{\alpha e_i e_j} \\
&= (C - (e_i e_j)S)(cte_0 + xe_1 + ye_2 + ze_3)(C + (e_i e_j)S) \\
&= (C - Se_i e_j)cte_0(C + Se_i e_j) + \\
&\quad (C - Se_i e_j)xe_1(C + Se_i e_j) + \\
&\quad (C - Se_i e_j)ye_2(C + Se_i e_j) + \\
&\quad (C - Se_i e_j)ze_3(C + Se_i e_j) \\
&= (C^2 cte_0 - SCcte_i e_j e_0 + SCcte_0 e_i e_j - S^2 cte_i e_j e_0 e_i e_j) + \\
&\quad (C^2 xe_1 - SCxe_i e_j e_1 + SCxe_1 e_i e_j - S^2 xe_i e_j e_1 e_i e_j) + \\
&\quad (C^2 ye_2 - SCye_i e_j e_2 + SCye_2 e_i e_j - S^2 ye_i e_j e_2 e_i e_j) + \\
&\quad (C^2 ze_3 - SCze_i e_j e_3 + SCze_3 e_i e_j - S^2 ze_i e_j e_3 e_i e_j)
\end{aligned} \tag{20}$$

At this point, we can see that there is symmetry. Due to this fact, two arbitrary values for i and j will be chosen here, and calculations will be continued with those. We will chose $i = 1$ and $j = 2$ to represent a rotation in the e_1, e_2-plane.

But first we can look at the terms seperately. We see that each term consists of a scalar (we will assume the coordinate variables x, y, z, ct to be scalars) multiplied with one of the base vectors $(s \cdot e_p)$. If we take the elaborated term above, for $(p \neq i, p \neq j)$, we see that it is of this form:

$$(C^2 se_p - SCse_i e_j e_p + SCse_p e_i e_j - S^2 se_i e_j e_p e_i e_j) \tag{21}$$

Furthermore (the bold fonts being added just to place emphasis; there is no conceptual or physical difference between e_p and $\mathbf{e_p}$)

$$\begin{aligned}
SCs\mathbf{e_p} e_i e_j &= -SCse_i \mathbf{e_p} e_j = SCse_i e_j \mathbf{e_p} \\
-S^2 se_i e_j e_p \mathbf{e_i} e_j &= -S^2 se_i \mathbf{e_i} e_j e_p e_j = S^2 se_j e_p e_j = S^2 se_p
\end{aligned} \tag{22}$$

Substitution into equation 21 yields:

$$(C^2 se_p + S^2 se_p - SCse_i e_j e_p + SCse_i e_j e_p) = se_p \tag{23}$$

It is not difficult to see that this is analogous to the multiplication of scalar-bivector complex conjugates, expressed in equation 18.

This can be understood, since a vector e_p is multiplied by two scalar-bivectorcomplex numbers, which are each others scalar-bivectorcomplex conjugates. We could visualise this effect as the multiplication of a "normal" complex number $e^{i\theta}$ by another complex number $e^{i\gamma}$. The result is, of course, $e^{i(\gamma+\theta)}$. This multiplication corresponds to rotating the vector that represents $e^{i\theta}$ in the complex plane by an angle γ. Similarly, the rotation that is effected in this case does not affect the coordinate axes of the other directions than those in which the rotation takes place.

In other words, the coordinates which are not found in the exponent of $(e^{\frac{\alpha}{2}e_ie_j})$ are not affected by the transformation. This simplifies our example.

Taking $i = 1$ and $j = 2$, and using our new information, we find the following, and use similar elimination techniques as used in equation 22:

$$\begin{aligned} x'e_1 + y'e_2 &= (C^2xe_1 - SCxe_1e_2e_1 + SCxe_1e_1e_2 - S^2xe_1e_2e_1e_1e_2) + \\ &\quad (C^2ye_2 - SCye_1e_2e_2 + SCye_2e_1e_2 - S^2ye_1e_2e_2e_1e_2) \\ &= (C^2xe_1 - SCxe_2 - SCxe_2 - S^2xe_1) + \\ &\quad (C^2ye_2 + SCye_1 + SCye_1 - S^2ye_2) \\ &= (C^2x + 2SCy - S^2x)e_1 + (C^2y - 2SCx - S^2y)e_2 \end{aligned}$$

And now substituting the equations for the sine and the cosine ($\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ and ($2\sin\alpha\cos\alpha = \sin 2\alpha$) and extracting the $\frac{1}{2}$ from α again, we arrive at the following equations:

$$\begin{aligned} ct' &= ct \\ x' &= x\cos\alpha + y\sin\alpha \\ y' &= -x\sin\alpha + y\cos\alpha \\ z' &= z \end{aligned} \tag{24}$$

This result corresponds to a 4×4 matrix equivalent of the standard 2×2 rotation matrix. But a further discussion of this matrix would leave the road towards answering the main question of this paper. Obviously, the above derivation can be done for any nonzero and non-equal i and j and it shows that this derivation is a general phenomenon. Furthermore, the identity thus arrived at bears a striking resemblance with the equations for the Lorentz Transformation in rotations we started out with in equation 1.5. In fact, it is apparent that $\frac{\alpha}{2} = \theta$ and this makes the equations equivalent.

2.4 Lorentz Boosts

And in the case $i > 0, j = 0$, the derivation up to and including equation 20 holds, except for the fact that then we of course define $C \equiv \cosh\alpha$ and $S \equiv \sinh\alpha$. The symmetry then of course still holds, therefore an arbitrary number can be assigned to i, which in this example will be 1. Furthermore, for $p > 0, p \neq i$, like in equation 23 it holds that $-S^2 se_i e_0 e_p e_i e_0 = -S^2 se_i e_p e_i = -S^2 se_p$ and $\cosh\alpha - \sinh\alpha = 1$. Therefore:

$$(C^2 se_p - SCse_i e_0 e_p + SCse_p e_i e_0 - S^2 se_i e_0 e_p e_i e_0) = se_p \tag{25}$$

This, again, corresponds to the result obtained in equation 19, and implies the conclusion that the non-$i, 0$-coordinates are unaffected by a boost.

Therefore the analogous form of equation 20 can be reduced to:

$$\begin{aligned} ct' e_0 + x' e_1 &= (C^2 cte_0 - SCcte_1 e_0 e_0 + SCcte_0 e_1 e_0 - S^2 cte_1 e_0 e_0 e_1 e_0) + \\ &\quad (C^2 xe_1 - SCxe_1 e_0 e_1 + SCxe_1 e_1 e_0 - S^2 xe_1 e_0 e_1 e_1 e_0) \\ &= (C^2 cte_0 - SCcte_1 - SCcte_1 + S^2 cte_0) + \\ &\quad (C^2 xe_1 - SCxe_0 - SCxe_0 + S^2 xe_1) \\ &= (C^2 ct - 2SCx + S^2 ct)e_0 + (C^2 x - 2SCct + S^2 x)e_1 \\ &= (ct \cosh 2\alpha - x \sinh 2\alpha)e_0 + \\ &\quad (x \cosh 2\alpha - ct \sinh 2\alpha)e_1 \end{aligned}$$

Which is true, since:

$$\begin{aligned} \cosh^2\alpha + \sinh^2\alpha &= \cosh 2\alpha \\ 2\sinh\alpha\cosh\alpha &= \sinh 2\alpha. \end{aligned} \tag{26}$$

We can summarise this result as follows (extracting the $\frac{1}{2}$ from α):

$$\begin{aligned} ct' &= ct\cosh\alpha - x\sinh\alpha \\ x' &= -ct\sinh\alpha + x\cosh\alpha \\ y' &= y \\ z' &= z \end{aligned}$$

which answers our main question.

3 Conclusions

In conclusion, it has been shown in this paper that the initial formulation of the Lorentz Transformation – in the way it is found in most textbooks – is equivalent to the Geometric Algebra in the sense that the coordinate transformation has the same effect, but is different in the sense that Geometric Algebra enables the physicst to choose a coordinate system at a later stage, which is preferable in most applications.

It has been shown too that the quantity $e^{\frac{\alpha}{2}\varepsilon_i}$ can be written as a sum of cosines and sines if $i > 0$ or hyperbolic sines and cosines if $i = 0$ and, in sofar as it can be understood as a scalar-bivector complex number, is analogous to "conventional" complex numbers in their behaviour.

References

[CW98] Greenwood D.A. Cottingham W.N. *An Introduction to the Standard Model of Particle Physics.* Cambridge University Press, Cambridge, 1998.

[Sch72] Melvin Schwartz. *Principles of Electrodynamics.* McGraw-Hill Book Company, New York, 1972.