The Problem of Particular Affirmitives in Leibniz's Original Formulation of Logical Calculus

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Abstract

This paper will discuss the problem of the incorporation of particular affirmative propositions in Leibniz's system of logical calculus, which was designed in such a way that all inferences could be arithmetically tested. First of all, some of the definitions of the concepts that will have to be used in this discussion will be clarified. Secondly, Leibniz's thoughts on formal transcription of logical inferences and on the basis of his work *Elements of a Calculus* (1679) the system of logical calulus that derives from his transcription will be discussed. It will become clear that in Leibniz's original formulation the problem of particular affirmative propositions lies in the case in which not one of the two terms of a particular affirmative proposition is the genus of the other, on which assumption the assignment of products of prime numbers to compound terms relied. This problem directly reflects the inability of integers to model conceptual complexity and reflects the disrepancy between the intensional and extensional approaches to the problem.

1 Introduction

If one is optimistic that there is a unique definitive truth, then perhaps there is a way in which man can determine the truth of each proposition beyond doubt. The question would be what criterion would be suited to this purpose and the clarity, unambiguity and certainty of mathematics renders it a virtuous example of a system of definitive truth.

These considerations led the philosopher Leibniz to his attempts to firstly construct a convention in logical symbolic transcription, and then design a system, fundamentally defined by arithmetic, by which the validity of inferences could be verified through simple calculations.

Leibniz writes [*Logical Papers*, 18]:

> This is our prerogative: that by means of numbers we can judge immediately whether propositions presented to us are proved, and that what others could hardly do with the greatest mental labour and good fortune, we can provide with the guidance of symbols alone, by a sure and truly analytical method. As a result of this, we shall be able to show within a century what many thousands of years would hardly have granted to mortals otherwise.

1.1 Problem

The main question that will be addressed in this paper is:

Main question What is the origin of the problem of particular affirmitives in Leibniz's original formulation of the system of logical calculus?

2 Preliminary Analysis

2.1 Terminology

Before embarking on discussing the main question it seems wise to define some of the concepts that will be used and briefly discuss each of them – which is what shall be done in this section.

First of all, by *"term"* will be understood here, a word or a collection of words that (1) does not exclusively perform a grammatical function, and (2) can be understood in itself without any reference to other words usually appearing in context. For instance, it seems reasonable to assume that "or" is not a term (since it is solely grammatical, and, therefore, cannot be understood in itself, but only in relation with terms in its grammatical vicinity) but "Socrates" (a name) is.

Terms can appear in several modes, which can, for now, generally be divided into *noun*, *verb* and *adjective*. By means of exclusion of grammatical words and particles however, these do not distinguish between the applications of terms, but are rather synonymous. For instance, one can say "Socrates is a thinker," or, "Socrates is thinking," or "Socrates thinks," using the nominal, adjectival and verbal forms, respectively, of the concept of "thinking." However, the two sentences do not differ in meaning significantly enough to distinguish them in the logical analysis within the framework of this paper. This, therefore, is an assumption.

Furthermore, terms can refer to individuals on the one hand, in which case they will be called *singular*, or to groups of individuals on the other hand, grouped by some relevant criterion that is understood to be in the term and corresponds to a universal notion, in which case they will be called *universal*. The precision of this definition will be of importance to the distinction between intension and extensino.

A *proposition* consists of terms and grammatical connection words, and serves to define a certain conceptual relationship between the terms. In the simple situation that will be found at the starting point for this analysis, a proposition consists of only two terms: a subject and a predicate. The predicate is what is said of the subject, that is, what is predicated of the subject.

Socrates	is	a thinker.
subject	(grammatical)	predicate

2.2 Intension/Extension and Species/Genera

In philosophy of language the concepts intension and extension express a distinction between ways to consider the meaning of a term. By extension, or denotation, is understood the collection of individual things to which the term applies. By intension, or connotation, is understood the set of qualities that those things that correspond to the term are presumed to have in common. For instance, the intension of the term "table" is a piece of furniture, consisting in a plate and, typically, four legs, and the extension are the so many tables that are found in reality.

The distinction of species and genera here will be that used by Artistotle and adopted by Leibniz, and is best understood extensionally; both of them refer to groups of individuals and all the individuals of the species are included in the genus, but not vice versa; in addition to the individuals of the species there are found many other individuals comprehended in the genus. For instance, "furniture" can be said to be a genus in relation to the species of "tables."

In the closing of the preliminary discussion on intension and extension, it seems noteworthy that there is what will be referred to as the *quantitatively inverse relationship* between the intension and the extension in the sense that the more is comprehended in the one, the less is comprehended in the other. For instance, moving attention from "man-made equipment", via "furniture" to finally "tables," is moving from genera to species, and thereby (1) shifting extensionally from an enormous set of individuals comprehended by "man-made equipment" to a smaller and smaller included sets, but at the same time (2) shifting intensionally from a very general concept to concepts including many different and distinct features. In general, it will be held here that the more is comprehended intensionally in a term, the less is comprehended extensionally, and therefore also vice versa.

It is this inverse relationship that causes the fact that, properly speaking, extensionally the species is in the genus, but intensionally the genus is in the species (all general that constitute the concept of the genus must be included in the species, together with additional features that set the members of the set of the species apart from the other members of the genus).

2.3 Aristotle's Inference Schemes

Aristotle designed an exact system of inference and distinguished different propositions, which are illustrated below, together with their current conventional formal notation:

A	universal affirmative	All S is P	$\forall x(Sx \rightarrow Px)$
E	universal negative	No S is P	$\forall x(Sx \rightarrow \neg Px)$
I	particular affirmative	Some S is P	$\exists x(Sx \wedge Px)$
O	particular negative	Some S is not P	$\exists x(Sx \wedge \neg Px)$

Firstly, it is important to note that there can be identified are mutually exclusive pairs, such that one of each pair can be formulated in terms of a negation of the complementary case. For instance, It seems reasonable to hold as logically equivalent on the one hand that all S are P, and on the other, that there is no S which is not P. This can be expressed as follows:

$$A(S,P) = \neg O(S,P) \qquad , \qquad E(S,P) = \neg I(S,P) \tag{1}$$

Secondly, the propositions have been worded in such a way that they can be interpreted both intensionally and extensionally; intensionally both are qualities and extensionally both are sets of individuals and in each case the relations of complementarity expressed above can be understood.

2.4 Truth

When assigning symbols to propositions, terms, or relations between them, it becomes necessary to consider the nature of these symbols and the notion of truth that they rely on or judge upon.

As to the definition of truth, Leibniz prefers to consider truth as, generally, that case in which the concept of the predicate lies in that of the subject. This is also known as *praedicatum inest subjecto*. The intensional nature of this definition is apparent in the following quote. Leibniz, referring to a similar conception of truth found in Aristotle, writes [*New Essays on Human Understanding*, 486 (Bassler 118)]:

> This manner of statement deserves respect; for indeed the predicate is in the subject, or rather the idea of the predicate is included in the idea of the subject.

From this assumption, Leibniz built his system of logical calculus. As to the nature of symbols, Leibniz wrote [*1677 Dialogue* (Bassler 118)]:

> There is some relation or order in the characters which is also in the things, especially if the characters are well invented. [...] [T]heir use and connection have something which is not arbitrary, namely a definite analogy between characters and things, and the relations which different characters expressing the same thing have to each other. This analogy or relation is the basis of truth.

It then is evident that Leibniz considers correspondence between the relations between the terms on the one hand, and the relations between the "things" they refer to as the basis for truth in the relations.

2.5 Prime Numbers

Since Leibniz included the notions of prime and relative prime numbers in his logical calculus, a small exploration of this subject will increase understanding of the way Leibniz attemped to model conceptual relations.

A *factor* is a number and any number is said to have a particular factor if it can be divided by it, yielding an integer number. Since we can write 6 as $6 = 2 \times 3$, we can say that 2 and 3 are factors of 6, and 2 is not a factor of 5, since $\frac{5}{2}$ is not an integer number.

A number is said to be an (absolute) *prime number* when its only factors are the number itself and 1. For instance, 2 and 5 are a prime numbers, but 6, as we saw above has 2 and 3 as factors, and thus has other factors than itself and 1 and therefore is not a prime number.

A *common factor* of two numbers a and b is a number that is a factor for both a and b. Two numbers are said to be *relative prime numbers* if they only have 1 as a common factor. For instance, 3 and 4 are relative prime numbers, but 4 and 6 are not, since they have the common factor 2.

It is not difficult to see that any two different absolute prime numbers are also relative prime numbers, since prime numbers have only themselves and 1 as divisors, and therefore two unequal prime numbers have no common factors but 1.

3 Historical Development of Leibniz' Logical System

Leibniz developed his attempt to generate a convention in logical symbolic transcription over the years, and this resulted in the three versions of it which are currently generally distinguished. In this section, only the first will be discussed, and the different interpretations that Leibniz gave for it will be addressed.

In his article "Leibniz's Interpretation of His Logical Calculi", Nicolas Rescher establishes the five building blocks of the logical theory:

1. Variables for terms
2. Singular and binary operators
3. Relations between terms
4. Rules of Inference
5. A group of axiomatic statements

3.1 First Formulation

The first formulation was designed by Leibniz around 1679 [Rescher, 3]. Variables for terms are simply the lower-case letters a, b, etc. The singular operator is the negation ($\neg$) and the binary operator is juxtaposition (denoted by simply writing the two terms one after another). Relations between terms are "est" ($\subset$), equality ($=$), and inequality ($\neq$). Leibniz then defined the group of axiomatic statements by means of these words and they are the following [Rescher, 3]:

1	$a \subset a$
2	$a = \neg\neg a$
3	$a \subset b \iff \neg b \subset \neg a$
4	$(a \subset b) \wedge (b \subset c) \Rightarrow a \subset c$
5	$(a \subset b) \wedge (b \subset a) \iff a = b$
6	$a \neq b \iff \neg(a = b)$
7	$(a = b) \Rightarrow (b = a)$
8	$(a = b) \wedge (b = c) \Rightarrow (a = c)$
9	$a = aa$
10	$ab = ba$
11	$a \subset bc \iff (a \subset b) \wedge (a \subset c)$
12	$(a \subset b) \Rightarrow (ca \subset cb)$
13	$(b \subset a) \wedge (c \subset a) \Rightarrow (bc \subset a)$
14	$(a \subset b) \wedge (c \subset d) \Rightarrow (ac \subset bd)$
15	$ab \subset a$
16	$ab \subset b$
17	$\mathrm{proper}(a) \Rightarrow \neg(a \subset \neg a)$
18	$\mathrm{proper}(a) \Rightarrow ((a \subset \neg b) \Rightarrow \neg(a \subset b))$

Before continuing, it is fruitful to note that the expressions 5 and 6 define the binary operators $=, \neq$. To come to an understanding of this logical system it therefore would suffice to, as Rescher notes [5]:

> specify (1) the set of "terms," (2) the effects of the *non* operator, and of the operation of juxtaposition, (3) the meaning of the relation "est", and (4) the meaning of propriety; provided that this is done so as to satisfy the assertions.

Leibniz himself gave two alternative interpretations for this first system.

3.2 Intensional Interpretation

Terms are the intensions of predicates, in other words, collections of properties. *Non-x* refers to not having the property x. Juxtaposition refers to having both properties (ab means having both a and b). Furthermore, Rescher notes that [6] "[a] term is proper if it is not of universal extension (i.e., null extension)." Also, $a \subset b$ is to say that a contains in its understanding b.

3.3 Extensional Interpretation

Terms are considered sets of objects. *Non-x* refers to all objects not belonging to set x. Juxtaposition refers to the set of objects that are in both sets juxtaposed. Any set that is not empty is proper. $a \subset b$ in this interpretation means that a is fully contained in the b.

3.4 Comparison

First of all, all expressions mentioned as axioms can be understood as meaningful in both interpretations. However, the fact that one and the same formally defined system can be understood in both ways seems quite plausible and satisfying, but the inverse relationship between intension and extension pointed out in section 2.2 demands caution. Indeed, when $a \subset b$ would by the intensional interpretation mean that "a includes b" in its understanding, therefore b is a genus relative to the species a, and therefore extensionally one should say "a is included in b." This discrepancy of terms causes much confusion and obstructs interpretation of Leibniz's logical calculus.

Indeed, when discussing species and genera, Leibniz uses his intensional interpretation and foresees the existence of problems [Bassler, 121]:

> I consider the genus as a part of the species, since the concept of the species is produced from the concept of the genus and of the differentia. On this principle I constructed this method of calculation, since I considered ideas and not individuals. However, proceeding in this way it was very difficult to descend from the genus to the species, since it is a progress from the part to the whole.

4 Leibniz's Logical Calculus

4.1 Number assignment to concepts

In his essay *"Elements of a Calculus"* (1679), Leibniz elaborates how calculus could be performed on logical inferences.

Leibniz starts out by asserting that to any term a number can be assigned, and that that will be enough to capture that part of its meaning that is relevant to the analysis by logical calculus. Leibniz holds that this fact reflects a sort of mathematical certainty in our concepts [17]:

> For the moment, however, numbers are of the greatest use, because of their certainty and of the ease with which they can be handled, and because in this way it is evident to the eye that everything is certain and determinate in the case of concepts, as it is in the case of numbers.

Prime numbers will be used in this assignment.

Furthermore, in the simple case where two concepts a and b are combined in another concept, the number assigned to this concept is the (arithmetic) product ab, analogous to the logical system introduced in subsection 3.1 [18]:

> when the concept of a given term is composed directly of the concepts of two or more other terms, then the symbolic number of the given term should be produced by multiplying together the symbolic numbers of the terms which compose the concept of the given term. For example, since man is a rational animal, if the number of animal a, is 2, and of rational r, is 3, then the number of man, h will be the same as ar: in this example, 2×3, or 6.

It is important to note at this point that Leibniz proceeds from the intension of the concepts.

4.2 Analysis of fundamentals

First of it can be noted here that Leibniz hoped to build all concepts present to the human mind from a finite number of simple and indivisible ones; these being assigned prime numbers to represent their conceptual indivisibility. Secondly, Leibniz is optimistic that this will enable us to cover all concepts in the world. Even though perhaps his intuition was correct, it is not immediately evident that all concepts of the world can be placed in a arithmetically specified hierarchical order. It is not the problem of *regressus ad infinitum* which seems most pressing here – since any concept could be taken as a starting point and consequently others could be derived from it – but rather circularity in definitions might prevent a complete arithmetical rendering the conceptual complexity that is present to the human intellect. However, since this is not the main topic of this paper, a further discussion of will be omitted. Leibniz writes [18]:

> The rule given in article 4 is sufficient for our calculus to cover all things in the whole world, as far as we have distinct concepts of them, i.e. as far as we know some of their requisites by which, after we have examined them bit by bit, we can distinguish them from all others.

Particularly illuminating is the comment of O. Bradley Bassler in his article *Leibniz on intension, extension and syllogistic inference*, stating that a project of this sort can be thought of in the weak and the strong sense. In the weak sense, it would be a project of finding ways to evaluate inferences in logic, to which end arbitrary characteristic numbers would be assigned to concepts. In the strong sense, it would be finding a unique numerical characterisation for concepts, that would not only be used in any particular application, but could be applied in any situation. Bassler suggests that Leibniz did succeed considerably in the weak sense, but did not get very far in the strong sense.

4.3 Relations of terms

Leibniz then distinguishes several ways in which two terms can relate to each other:

- **Contained** – One is contained in the other.
  - **Coincident** – They are equivalent.
  - **Genus and Species** – One contains the other but they do not coincide.
- **Disparate** – Neither contains the other.
  - **Conspecies** They have something in common.

– **Heterogeneous** What they have in common is very remote.

Leibniz continues the elaboration of his system for logical calculus by asserting that any universal affirmative statement can be tested if the characteristic numbers assigned to each of the concepts are known, by simply verifying that the arithmetic corresponds. For instance, if "man is a rational animal" is characterised by $6 = 2 \times 3$ then it can be verified whether "all men are rational" is true, by checking whether being rational is included in the concept of men: dividing 6 by 2, $\frac{6}{2} = 3$ results an integer number, therefore the proposition is true [22]:

> From this, therefore, we can know whether some universal affirmative proposition is true. For in this proposition the concept of the subject, taken absolutely and indefinitely, and in general regarded in itself, always contains the concept of the predicate.

4.4 Problem of particular affirmatives

In particular affirmative propositions, this strategy cannot be applied, since particular affirmative propositions can be true without their universal affirmative counterparts to be true, therefore there is a need for a criterion that leaves open the possibility of the predicate not applying to all the instances of that which the subject refers to, but only some particular ones. Leibniz notes that [23]:

> it is not necessary that the predicate should be in the subject regarded in itself and absolutely; i.e. that the concept of the subject should in itself contain the concept of the predicate; it is enough that the predicate should be contained in some species of the subject, i.e. that "the concept of some instance or species of the subject should contain the concept of the predicate," even though it is not stated expressly what the species is.

This problem can be illuminated by considering the situation extensionally. Assuming one set of individuals x and another set of individuals y, such that, in the formal transcription we have used so far $x \subset y$. Now if $x \neq y$ and $x \neq \emptyset$ the universal affirmative claiming that $y \subset x$ ("All y have property A") is false, but the particular affirmative claiming that "Some y are A" is true (and those y must be x). As we have considered x as a subset of y, x can be considered a species and y the genus to which it belongs. Furthermore, Leibniz notes that, in the symbols introduced here, if $x = y$ instead of $x \subset y$ it could have been claimed universally ($y = x$), which follows immediately from axiom 7 [20].

Before going on with a discussion of the problem that follows, it should be clear that once this problem is resolved, the system of logical calculus can be freely applied to all of the four general propositions introduced in section 2.3, since, as has been shown, the particular negative statements can be understood in terms of their universal affirmative counterparts, and vice versa.

4.5 Analysis

Generally, there can be identified a number of sufficient (but not necessary) conditions for a particular affirmitive proposition ("Some S are P") to hold – for the subject S and the predicate P such that $S \wedge P \neq \emptyset$, most easily interpreted as properties –:

- $P \subset S$, $S \neq P$
- $S = P$
- $S \subset P$, $S \neq P$

Leibniz writes this intensionally [23]:

> [The concept of subject] will be related [to that of the predicate] as whole to part when the concept of the predicate, as genus, is in the concept of the subject, as species: e.g. if "bernicle" is the subject and "bird" the predicate. It will be related as whole to coincident whole when two equivalents are stated of each other reciprocally, as when "triangle" is the subject and "trilateral" the predicate. Finally, it will be related as part to whole, as when "metal" is the subject and "gold" the predicate.

It is apparent that in each of these cases, the particular affirmative proposition can be shown correct by arithmetic – that is, writing out the prime factors of the numbers assigned to concept of the predicate and the subject, and thereby verifying that they are not relative primes, will be sufficient to show that the particular affirmative proposition holds.

The problem of particular affirmative lies in the cases that fall under neither of the categories above, but still are sufficient for a particular affirmative proposition to hold. Leibniz distinguishes the following cases:

- A species of the subject contains the predicate
  - Coincident $\Rightarrow$ The predicate is a species of subject
  - Genus and Species $\Rightarrow$ The predicate is a genus of a species of the subject

Leibniz writes [24]:

> Now, two genera of the same species either coincide or, if they do not, they are necessarily related as genus and species. This is easily shown, since the concept of the genus if formed simply by casting-off [*abjectio*] from that of the species; since, therefore, from a common species of two genera, genera will appear on both sides by continued casting-off (that is, they will be left behind as superfluous concepts are cast off), one will appear before the other, and so one will seem to be a whole and the other a part.

Here it becomes evident how fundamental the problem of species/genera and intension/extension is. Leibniz proceeds by conceptually eliminating specific features of a species and thereby arriving at the concept of a genus. However, Leibniz assumes here that there is a unique order in which this elimination of concepts should proceed. For instance, if from the concept of an antique table, the features that correspond to its being a table would be cast off, the concept of antique object would remain. If, however, the features that correspond to its being an antique object would be cast off, the concept of a table would remain. From this example, it seems evident that two genera of the same species do not have to either coincide or be related as genus and species.

Leibniz recognises this and writes [24]:

> So we have a paralogism, and with it there falls much that we have said hitherto; for I see that a particular affirmative proposition holds even when neither term is a genus or species, such as "Some animal is rational."

We also see the intensional approach of Leibniz in this extract. Formulated extensionally, the problem ultimately is that the two sets S and P can have common elements $S \wedge P$ without $S \subset P$ or $P \subset S$. Therefore, it is not necessary that the concept of the subject includes the predicate (as is the case with universal affirmatives). But then the way of logical calculus, as it has proceeded up to this point, cannot be properly applied, since neither of the two terms need to have a common factor, i.e. they can be relative primes and the particular affirmative proposition can still hold.

5 Conclusion

The main question is thus answered; the problem of species and genera is the problem of particular propositions. In Leibniz's original formulation the problem of particular affirmative propositions lies in the case in which not one of the two terms of a particular affirmative proposition is the genus of the other, as a species. That, however, was the condition by which logical calculus could work by the assignment of prime numbers by products to compound terms. This problem directly reflects the inability of integers to model conceptual complexity and the discrepancy between intension and extension.

Leibniz closes his *Elements of a Calculus* with the words:

> Hence it is also evident that it is not necessary that the subject can be divided by the predicate or the predicate by the subject, on which we have so far built a great deal. What we have said, therefore, is more restricted than it should be; so we shall begin again.

6 References

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